PLANE WAVE DISCONTINUOUS GALERKIN METHODS

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Standard low order Lagrangian finite element discretization of boundary value problems for the Helmholtz equation $-\Delta u - \omega^2 u = f$ are afflicted with the so-called pollution phenomenon: though for sufficiently small $h\omega$ an accurate approximation of u is possible, the Galerkin procedure fails to provide it. Attempts to remedy this have focused on incorporating extra information in the form of plane wave functions $\boldsymbol{x} \mapsto \exp(i\omega\boldsymbol{d}\cdot\boldsymbol{x})$, $|\boldsymbol{d}| = 1$, into the trial spaces. Prominent examples of such methods are the plane wave partition of unity finite element method of Babuska and Melenk [1], and the ultra-weak Galerkin discretization due to Cessenat and Despres [3, 4]. Both perform well in computations, see the articles by Monk and Hutunen [5–7] for computational results for the ultra-weak approach.

It turns out that the latter method can be recast as a special so-called discontinuous Galerkin (DG) method employing local trial spaces spanned by a few plane waves. This perspective paves the way for marrying plane wave approximation with many of the various DG methods developed for 2nd-order elliptic boundary value problems. We have pursued this for a special primal DG method which generalizes the ultra-weak scheme.

For these methods we have developed a convergence analysis for the h-version, which achieves convergence through mesh refinement. Key elements are approximation estimates for plane waves and sophisticated duality techniques. The latter entail estimating how well local plane waves can approximate the solution of a dual problem. Unfortunately, we cannot not help invoking general polynomial estimates in Sobolev spaces for this purpose. This incurs unsatisfactory pollution-affected estimates that predict the onset of asymptotic convergence provided that $\omega^2 h$ is sufficiently small. However, these estimates seem to be sharp and hint at the disappointing fact that plane wave approaches offer no remedy for numerical dispersion, which, as is conjectured, inevitable haunts local volumetric discretizations for wave propagation problems [2].

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