Legendre Pseudospectral Method for the Incompressible Navier-Stokes Equations on the Sphere *

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Abstract

The Legendre pseudospectral approximation for numerical solution of the time-dependent incompressible Navier-Stokes equations on a spherical surface is presented. The fully discrete Legendre pseudospectral scheme is constructed. The stability of the scheme is analyzed and the convergence is proved.

Key words: Legendre pseudospectral method; incompressible Navier Stokes equations; stability; convergence.

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1 Introduction

The incompressible Navier-stokes equations on a spherical surface is the following nonlinear partial differential equations [1]:

$$\begin{cases} \dfrac{\partial U}{\partial t} + (U \cdot \nabla)U - \nu \triangle U + \nabla P = f, & \text{in } S \times (0, T], \\ \nabla \cdot U = 0, & \text{in } S \times (0, T], \\ U|_{t=0} = U_0, & \text{on } S, \end{cases} \tag{1.1}$$

where $U = (U^1, U^2)$ is the velocity vector, P is the ratio of the pressure to constant density, and ν is the kinematic viscosity coefficient. Let $S = \left\{(\lambda, \theta) : \; 0 \le \lambda < 2\pi, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$ be the unit spherical surface, where λ and θ are longitude and latitude coordinates on the spherical surface, respectively. It is naturally assumed that all functions in (1.1) have the period 2π for the variable λ. Furthermore they are regular at $\theta = \pm\frac{\pi}{2}$ and their first order derivatives with respect to λ vanish at the two poles.

We note that the velocity U and the pressure P in the above equations are coupled together by the incompressibility constraint $\nabla \cdot U = 0$, which makes the system difficult to solve numerically. Popular strategies to overcome this difficulty are, among others, the artificial compressibility method, the pressure stabilization method and the projection method [2, 3, 4, 5, 11].

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Since the spectral method has convergence rate of "infinite" order, it has become one of the most powerful tools for the numerical solution of nonlinear partial differential equations arising in fluid dynamics [6, 7]. Many papers have also attempted to use spectral methods for solving problems in spherical coordinates [8, 9, 10] and achieved satisfactory results. However, relatively less work has been done in numerical analysis of these spectral methods which use the function space and approximate subspaces in spherical coordinates. In this paper, we present a legendre pseudospectral method for solving the time-dependent Navier-Stokes equations in the spherical polar coordinates.

An outline of the paper is as follows. In section 2, by using the spherical harmonic functions as the basis functions, the Legendre pseudospectral approximation on a sphere is described. In section 3, we list several lemmas related to the Legendre pseudospectral approximation. The stability and convergence of the proposed scheme are analyzed in section 4 and 5 respectively.

2 Legendre Pseudo-spectral Method

We are going to construct the pseudospectral scheme for (1.1). First we introduce some approximation subspaces and define an interpolation operator. For a non-negative integer $n \geq 0$, denote by $L_n(x)$ the nth degree Legendre polynomial defined on $x \in [-1, 1]$ and recall the orthogonality relation

$$(L_i(x), L_j(x)) = \frac{2}{2i+1}\delta_{ij}, \qquad \forall i, j \geq 0,$$

where $(f, g) = \int_{-1}^{1} f(x)g(x)dx$. Also recall that $L'_n(x)$ satisfies the recurrence relation

$$L'_n(x) = \sum_{k=0}^{n-1}(2k+1)L_k(x),$$

For an integer $m(|m| \leq n)$, the associated Legendre polynomials are defined as

$$\begin{aligned} L_{m,n}(x) &= \sqrt{\frac{(2n+1)(n-m)!}{2(n-m)!}}(1-x^2)^{m/2}\frac{d^m}{dx^m}L_n(x), && 0 \leq m \leq n, \\ L_{m,n}(x) &= L_{-m,n}(x), && -n \leq m \leq 0. \end{aligned}$$

Then the spherical harmonic functions $Y_{m,n}(\lambda, \theta)$ are

$$Y_{m,n}(\lambda, \theta) = \frac{1}{\sqrt{2\pi}}e^{im\lambda}L_{m,n}(\sin\theta), \qquad -n \leq m \leq n. \tag{2.1}$$

Let N be any positive integer that truncates the series. We define the trail function space for pseudo-spectral approximation as

$$\widehat{V}_N = \left\{ v : v = \sum_{n=0}^{N}\sum_{m=-n}^{n} \widehat{v}_{m,n}Y_{m,n}(\lambda, \theta) \right\}. \tag{2.2}$$

Furthermore, define V_N as the real subspace of $\widehat{V}_N$, and V_N^0 as the subspace of V_N, whose average on the spherical surface vanishes.

Next, we consider the interpolation from $C(S)$ onto V_N . Let $x_j (0 \le j \le N)$ be the $N+1$ roots of the legendre polynomial $L_{N+1}(x)$. Clearly, $x_j \in [-1,1]$. Let

$$w_j = \frac{1}{(1-x_j^2)[L'_{N+1}(x_j)]^2}, \qquad 0 \le j \le N,$$

which are the $N+1$ weights in the Legendre-Gauss quadrature formula associated with the $N+1$ roots. Define S_N as a set of grid points on S,

$$S_N = \left\{ (\lambda_l, \theta_j) : \lambda_l = \frac{2\pi l}{2N+1}, 0 \le l \le 2N;\ \theta_j = \sin^{-1} x_j, 0 \le j \le N \right\}. \tag{2.3}$$

Then we define the interpolation operator I_N, from $C(S)$ onto V_N, as follows

$$I_N v = \sum_{n=0}^{N} \sum_{m=-n}^{n} v_{m,n} Y_{m,n}(\lambda, \theta), \tag{2.4}$$

where

$$v_{m,n} = \frac{2\pi}{2N+1} \sum_{l=0}^{2N} \sum_{j=0}^{N} v(\lambda_l, \theta_j) w_j Y^*_{m,n}(\lambda_l, \theta_j),$$

and symbol "$*$" denotes complex conjugate of $Y_{m,n}$. Moreover, we introduce the discrete inner product $(\cdot,\cdot)_N$ as

$$(u,v)_N = \frac{2\pi}{2N+1} \sum_{l=0}^{2N} \sum_{j=0}^{N} w_j v(\lambda_l, \theta_j) v^*_{m,n}(\lambda_l, \theta_j), \qquad \forall u, v \in C(S). \tag{2.5}$$

Define the bilinear form

$$J(u,v) = (v \cdot \nabla) u + \frac{1}{2} (\nabla \cdot v) u \tag{2.6}$$

It is easy to verify that $(J(u,v),u) = 0$. To tackle the incompressibility constraint, we adopt the approach of artificial compressibility that is to approximate the incompressible condition by the equation

$$\beta \frac{\partial P}{\partial t} + \nabla \cdot U = 0, \tag{2.7}$$

where $\beta >> 1$ is the artificial compressibility factor. Finally, we consider the finite difference discretization in the temporal direction. Let τ be the time step. Define

$$R_\tau = \left\{ t = k\tau : 0 \le k \le \frac{T}{\tau} \right\},$$

and

$$v_t(\lambda,\theta,t) = \frac{1}{\tau}[v(\lambda,\theta,t+\tau) - v(\lambda,\theta,t)],$$
$$\widehat{v}(\lambda,\theta,t) = \frac{1}{2}[v(\lambda,\theta,t+\tau) + v(\lambda,\theta,t)],$$

A fully discrete Legendre pseudo-spectral scheme for solving (1.1) with the approach of artificial compressibility is to find$(u(t),p(t)) \in V_N \times V_N^0$ for all $t \in R_\tau$, such that

$$\begin{cases} u_t + I_N J(\widehat{u}(t), \widehat{u}(t)) + \nu \triangle \widehat{u}(t) + \nabla \widehat{p}(t) = \widehat{f}(t) \\ \beta p_t(t) + \nabla \cdot \widehat{u}(t) = 0, \\ u(0) = I_N U_0, \quad p(0) = 0. \end{cases} \tag{2.8}$$

3 Lemmas

We give several lemmas related to the Legendre pseudospectral approximation. Let $D(S)$ be the set of all infinitely differentiable function defined on S. The dual space of $D(S)$ is denoted by $D'(S)$. Define

$$L^2(S) = \{u \in D'(S): \ \|u\| < \infty\},$$

equipped with following inner product and norm

$$(u,v) = \int\int_S uv ds = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} u(\lambda,\theta)v(\lambda,\theta)\cos\theta d\theta d\lambda, \qquad \|u\| = (u,u)^{\frac{1}{2}}.$$

Also we define

$$H^1(S) = \left\{u : u, \frac{1}{\cos\theta}\frac{\partial u}{\partial\lambda}, \frac{\partial u}{\partial\theta} \in L^2(S)\right\}.$$

Its semi-norm and norm are respectively

$$|u|_1 = \left(\left\|\frac{1}{\cos\theta}\frac{\partial u}{\partial\lambda}\right\|^2 + \left\|\frac{\partial u}{\partial\theta}\right\|^2\right)^{\frac{1}{2}}, \qquad \|u\|_1 = (\|u\|^2 + |u|_1^2)^{\frac{1}{2}}.$$

For any real number $r \geq 0$, we define H^r as the complex interpolation between the two spaces $H^{[r]}(S)$ and $H^{[r+1]}(S)$. Since $Y_{m,n}$ are the eigenfunctions of the spherical Laplace operator $\triangle$, corresponding to the eigenvalues $n(n+1)$, the norm $\|u\|_r$ is equivalent to $\left(\sum_{n=0}^{\infty}\sum_{|m|\leq n}(n+1)^{2r}\widehat{u}_{m,n}^2\right)^{\frac{1}{2}}$, where $\widehat{u}_{m,n}^2$ being the Fourier coefficients related to the spherical harmonic functions $Y_{m,n}$. Besides, let $\|u\|_{r,\infty} = \|u\|_{C^r(S)}$, $\|u\|_\infty = \|u\|_{C(S)}$, etc. In order to derive the error estimates, we define the L^2 orthogonal projection operator P_N, i.e., for any $u \in L^2$, $P_N u \in V_N$ satisfies, $(u - P_N u, v) = 0, \ \forall v \in V_N$. Throughout this paper we shall use c to denote a general positive constant independent of τ and N. It can be different in different cases.

Lemma 1. *If* $0 \leq r \leq \beta$, *then for all* $u \in H^\beta(S)$,

$$\|u - P_N u\|_r \leq cN^{r-\beta}\|u\|_\beta$$

Proof.

$$\begin{aligned}
\|u - P_N u\|_r^2 &\leq c\sum_{m=-N}^{N}\sum_{n=N+1}^{\infty} n^r(n+1)^r|\widehat{u}_{m,n}|^2 + c\sum_{|m|>N}\sum_{n=|m|}^{\infty} n^r(n+1)^r|\widehat{u}_{m,n}|^2 \\
&\leq c\sum_{m=-N}^{N}\sum_{n=N+1}^{\infty} n^r(n+1)^r|\widehat{u}_{m,n}|^2 + c\sum_{|m|>N}\sum_{n=N+1}^{\infty} n^r(n+1)^r|\widehat{u}_{m,n}|^2 \\
&\leq cN^{2r-2\beta}\sum_{m=-\infty}^{\infty}\sum_{n=N+1}^{\infty} n^\beta(n+1)^\beta|\widehat{u}_{m,n}|^2, \\
&\leq cN^{2r-2\beta}\|u\|_r^2
\end{aligned}$$

□

Lemma 2. *(Inverse Inequality) If* $0 \leq r \leq \beta$, *then for all* $\forall u \in V_N$,

$$\|u\|_\beta \leq cN^{\beta-r}\|u\|_r$$

Proof. Let

$$u = \sum_{m=-N}^{N} \sum_{n=|m|}^{N} \widehat{u}_{m,n} Y_{m,n}(\lambda, \theta).$$

Since $Y_{m,n}(\lambda, \theta)$ is the eigenfunction of the operator $-\triangle$ on S, with eigenvalue $-n(n+1)$. Thus for any $u \in H^r(s)$, the norm is equivalent to

$$\left(\sum_{m=-\infty}^{\infty} \sum_{n \geq |m|} n^r (n+1)^r |\widehat{u}_{m,n}|^2 \right)^{\frac{1}{2}}$$

□

Lemma 3. [7] *If $u \in C(S)$ and $v \in V_N$, then:*

$$(i)\ I_N v = v, \qquad (ii)\ (I_N u, v) = (I_N u, v)_N = (u, v)_N.$$

Lemma 4. [5] *For all $u \in C(S)$ we have*

$$\|I_N u\| = \|I_N u\|_N \leq \|u\|_N,$$
$$\|u - I_N u\|_N = \inf_{\forall v \in V_N} \|u - v\|_N.$$

Lemma 5. [5] *For all $u = (u^{(1)}, u^{(2)}) \in H^1(S)$*

$$\left\| \frac{\partial u^{(1)}}{\partial \lambda} \right\| + \left\| \frac{\partial u^{(2)}}{\partial \lambda} \right\| \leq \|u\|_{H^1(S)},$$

Lemma 6. *Assume $0 \leq r \leq \beta$ and $\beta > 1$. Then for all $u \in H^\beta(S)$,*

$$\|u - I_N u\|_r \leq c N^{1+r+\varepsilon-\beta} \|u\|_\beta$$

where ε is an arbitrary small number.

Proof. First we consider the case $r = 0$. By the embedding theorem on spherical surface, we have

$$\|u\|_\infty \leq c\|u\|_{1+\varepsilon}, \qquad \forall \varepsilon > 0 \tag{3.1}$$

Thus $H^\beta(S) \subset C(S)$ and $I_N u$ is well defined. It follows from (ii) of Lemma 4 that

$$\|u - I_N u\| \leq \|u - P_N u\|_N \leq c\|u - P_N u\|_\infty,$$

Because $I_N u \in V_N$, $P_N u \in V_N$, we have from Lemma 3 that

$$\begin{aligned} \|u - I_N u\| &\leq \|u - P_N u\| + \|I_N u - P_N u\| = \|u - P_N u\| + \|I_N u - P_N u\|_N \\ &\leq \|u - P_N u\| + \|u - I_N u\|_N + \|u - P_N u\|_N \leq 3\|u - P_N u\|_\infty \end{aligned}$$

The combination of (3.1) and Lemma 2 leads to

$$\|u - P_N u\|_\infty \leq c\|u - P_N u\|_{1+\varepsilon} \leq c N^{1+\varepsilon-\beta} \|u\|_\beta,$$

where ε is an arbitrary small number. If $r > 0$, then we have from lemma 2 that

$$\begin{aligned} \|I_N u - P_N u\|_r &\leq c N^r \|I_N u - P_N u\| \\ &\leq c N^r (\|u - P_N u\| + \|u - I_N u\|) \\ &\leq c N^{1+r+\varepsilon-\beta} \|u\|_\beta, \end{aligned}$$

which, together with Lemma 1, implies the conclusion of this lemma. □

4 The Stability

Suppose that the initial values $u(0)$, $p(0)$ in (2.8) have error $\widetilde{u}_0$, $\widetilde{p}_0$ and that the right hand terms in the first and second equation have errors $\widetilde{f}(t)$ and $\widetilde{g}(t)$ respectively. Then the error $\widetilde{u}(t)$ and $\widetilde{p}(t)$ of $u(t)$ and $p(t)$ satisfy.

$$\begin{cases} \widetilde{u}_t(t) + I_N[J(\widehat{\widetilde{u}}(t), \widehat{u}(t) + \widehat{\widetilde{u}}(t)) + J(\widehat{u}(t), \widehat{\widetilde{u}}(t))] \\ \qquad -\nu\triangle\widehat{\widetilde{u}}(t) + \nabla\widehat{\widetilde{p}}(t) = \widehat{\widetilde{f}}(t), \\ \beta\widehat{\widetilde{p}}_t(t) + \nabla\cdot\widehat{\widetilde{u}}(t) = \widetilde{g}(t), \\ \widetilde{u}(0) = \widetilde{u}_0, \quad \widetilde{p}(0) = \widetilde{p}_0, \end{cases} \tag{4.1}$$

By taking inner product of the first equation of (4.1) with $2\widehat{\widetilde{u}}(t)$, and the second equation of (4.1) with $2\widehat{\widetilde{p}}(t)$, we get

$$\left(\|\widetilde{u}(t)\|^2 + \beta\|\widetilde{p}(t)\|^2\right)_t + 2\nu|\widehat{\widetilde{u}}(t)|_1^2 + F = 2(\widehat{\widetilde{f}}(t), \widehat{\widetilde{u}}(t)) + 2(\widetilde{g}(t), \widehat{\widetilde{p}}(t)), \tag{4.2}$$

where

$$F = 2\left(I_N J(\widehat{u}(t), \widehat{\widetilde{u}}(t)), \widehat{\widetilde{u}}(t)\right)$$

Now we estimate the inner product and F, clearly

$$2\left|(\widehat{\widetilde{f}}(t), \widehat{\widetilde{u}}(t))\right| \le c(\|\widetilde{u}(t)\|^2 + \|\widetilde{u}(t+\tau)\|^2 + \left\|\widetilde{f}(t)\right\|^2 + \left\|\widetilde{f}(t+\tau)\right\|^2)$$
$$2\left|(\widetilde{g}(t), \widehat{\widetilde{p}}(t))\right| \le c(\|\widetilde{p}(t)\|^2 + \|\widetilde{p}(t+\tau)\|^2 + \|\widetilde{g}(t)\|^2)$$

$$|F| \le c\,\|\widehat{u}(t)\|_{1,\infty}\left\|\widehat{\widetilde{u}}(t)\right\|_{H^1(S)}\left\|\widehat{\widetilde{u}}(t)\right\| \le \nu\left|\widehat{\widetilde{u}}(t)\right|_1^2 + \frac{c}{\nu}\|\widehat{u}(t)\|_{1,\infty}^2\left\|\widehat{\widetilde{u}}(t)\right\|^2.$$

Putting the above estimations in (4.2), we get

$$\begin{aligned} \left(\|\widetilde{u}(t)\|^2 + \beta\,\|\widetilde{p}(t)\|^2\right)_t + \nu\left|\widehat{\widetilde{u}}(t)\right|_1^2 \le{} & A(\|\widetilde{u}(t)\|^2 + \|\widetilde{u}(t+\tau)\|^2 \\ & + \|\widetilde{p}(t)\|^2 + \|\widetilde{p}(t+\tau)\|^2) + G(t) \end{aligned} \tag{4.3}$$

where

$$A = (1 + \frac{c}{\nu})\|u(t)\|_{1,\infty}^2$$
$$G(t) = c\left(\left\|\widetilde{f}(t)\right\|^2 + \left\|\widetilde{f}(t+\tau)\right\|^2 + \|\widetilde{g}(t)\|^2\right),$$

let τ be suitably small, and define,

$$E(t) = \|\widetilde{u}(t)\|^2 + \beta\,\|\widetilde{p}(t)\|^2 + \nu\tau\sum_{t'=0}^{t-\tau}\left|\widehat{\widetilde{u}}(t')\right|_1^2,$$

$$\rho(t) = 2(\|\widetilde{u}(0)\|^2 + \beta\,\|\widetilde{p}(0)\|^2) + \tau\sum_{t'=0}^{t-\tau} G(t')$$

Summing up (4.3) for $t' = 0, \tau, ..., t-\tau$, we have

$$E(t) \le \rho(t) + 2A\tau\sum_{t'=0}^{t-\tau} E(t')$$

By applying Grownwall's inequity, we assert the following theorem for stability.

Theorem 1. *Suppose τ is suitably small, then there exist a positive constant A, such that for all $t \in R_\tau$*

$$E(t) \le \rho(t) e^{2At}.$$

5 Convergence

In this section, we analyze the convergence of the scheme (2.8) and derive the rate of convergence. Suppose $U(t)$, $P(t)$ is the exact solution of (1.1). Let $U^N(t)$ and $P^N(t)$ be its L^2 projection onto V_N that is $U^N(t) = P_N U(t)$, $P^N(t) = P_N P(t)$. Then $U^N(t)$ and $P^N(t)$ satisfies

$$\begin{cases} \frac{\partial \widehat{U}^N}{\partial t}(t) + \frac{1}{2} P_N \left[(U(t) \cdot \nabla) U(t) + (U(t+\tau) \cdot \nabla) U(t+\tau) \right] - \nu \triangle \widehat{U}^N(t) = P_N \widehat{f}(t) \\ \beta \frac{\partial \widehat{U}^N}{\partial t}(t) + \nabla \cdot \widehat{U}^N(t) = 0, \\ U^N(0) = P_N U_0, \quad P^N(0) = 0 \end{cases} \tag{5.1}$$

Define

$$\widetilde{U}(t) = U^N(t) - u(t), \qquad \widetilde{P}(t) = P^N(t) - p(t),$$

we drive from (2.8) and (5.1)

$$\begin{cases} \widetilde{U}_t(t) + I_N[J(\widehat{\widetilde{U}}(t), \widehat{U}(t) + \widehat{\widetilde{U}}(t)) + J(\widehat{U}(t), \widehat{\widetilde{U}}(t))] + \nu \triangle \widehat{\widetilde{U}}(t) \\ \qquad + \nabla \widehat{\widetilde{P}}(t) = E_1(t) + E_2(t) + E_3(t) \\ \beta P_t(t) + \nabla . \widehat{\widetilde{U}}(t) = \beta E_4(t), \\ \widetilde{U}(0) = (P_N - I_N) U_0, \quad \widetilde{U}(0) = 0, \end{cases} \tag{5.2}$$

where

$$\begin{aligned} E_1(t) &= (P_N - I_N) \widehat{f}(t), \\ E_2(t) &= U_t^N(t) - \frac{\partial \widehat{U}^N}{\partial t}(t), \\ E_3(t) &= I_N J(\widehat{U}(t), \widehat{U}(t)) - \frac{1}{2} P_N \left[(U(t) \cdot \nabla) U(t) + (U(t+\tau) \cdot \nabla) U(t+\tau) \right], \\ E_4(t) &= P_t^N(t) - \frac{\partial \widehat{P}^N}{\partial t}(t), \end{aligned}$$

By an argument similar to that in the section 4, we get

$$\begin{aligned} \left(\left\| \widetilde{U}(t) \right\|^2 + \beta \left\| \widetilde{P}(t) \right\|^2 \right)_t + \nu \left| \widehat{\widetilde{U}}(t) \right|_1^2 &\le B(\left\| \widetilde{U}(t) \right\|^2 + \left\| \widetilde{U}(t+\tau) \right\|^2 \\ &\quad + \left\| \widetilde{P}(t) \right\|^2 + \left\| \widetilde{P}(t+\tau) \right\|^2) + H(t) \end{aligned} \tag{5.3}$$

where

$$B = (1 + \frac{c}{\nu}) \| U(t) \|_{1,\infty}^2 \qquad H(t) = \sum_{i=1}^{4} \| E_i(t) \|^2,$$

$$\widetilde{E}(t) = \left\|\widetilde{U}(t)\right\|^2 + \beta\left\|\widetilde{P}(t)\right\|^2 + \nu\tau\sum_{t'=0}^{t-\tau}\left|\widehat{\widetilde{U}}(t')\right|_1^2,$$

$$\widetilde{\rho}(t) = 2(\left\|\widetilde{U}(0)\right\|^2 + \beta\left\|\widetilde{P}(0)\right\|^2) + \tau\sum_{t'=0}^{t-\tau}H(t')$$

Summing up (5.3) for $t' = 0, \tau, ..., t-\tau$, we have

$$\widetilde{E}(t) \leq \widetilde{\rho}(t) + 2A\tau\sum_{t'=0}^{t-\tau}\widetilde{E}(t').$$

In order to get convergence rate for $\left\|\widetilde{U}(t)\right\|$ and $\left\|\widetilde{P}(t)\right\|$, we need only to estimate the order of $\widetilde{\rho}_1(t)$, then

$$\tau\sum_{t'=0}^{t-\tau}\|E_1(t')\|^2 \leq CN^{2(1+\varepsilon-r)}\|f(t)\|_r^2,$$

$$\tau\sum_{t'=0}^{t-\tau}\|E_2(t')\|^2 \leq CN^{2(1+\varepsilon-r)}\left\|\frac{\partial U}{\partial t}\right\|_{L^2(0,T;H^1(S))} + C\tau^4\left\|\frac{\partial^3 U}{\partial t^3}\right\|_{L^2(0,T;L^2(S))}^2,$$

$$\tau\sum_{t'=0}^{t-\tau}\|E_3(t')\|^2 \leq C\left(\||U|\|_r^2 + \||U^N|\|_r^2(S)\right) \times \left(N^{2(1+\varepsilon-r)}\|U\|_{C(0,T;H^r(S))}^2 + \tau^4\left\|\frac{\partial^2 U}{\partial t^2}\right\|_{L^2(0,T;H^r(S))}^2\right),$$

$$\tau\sum_{t'=0}^{t-\tau}\|E_4(t')\|^2 \leq C\tau^4\left\|\frac{\partial^2 P}{\partial t^2}\right\|_{L^2(0,T;L^2(S))}^2.$$

Consequently, we have

$$\widetilde{\rho}(t) \leq d\left(\tau^4 + N^{2(1+\varepsilon-r)}\right),$$

where d is positive constant depending only on ν and the norm of U and P in the spaces mentioned above. Finally we reach the following theorem for convergence rate.

Theorem 2. *Assume that the exact solution (U,P) of (1.1) satisfies the following smoothness $U \in H^1(0,T;H^1(S)) \bigcap H^3(0,T;L^2(S)) \bigcap C(0,T;H^r(S)) \bigcap H^2(0,T;L^2(S))$ $P \in H^2(0,T;L^2(S))$ $f \in C(0,T;H^r(S))$. Then there exist a positive constant d, such that for all $t \in R_\tau$*

$$\widetilde{E}(t) \leq d(\tau^4 + N^{2(1+\varepsilon-r)}),$$

where ε is a suitably small number.

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