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Equilibrium attractive properties of a class of multistep Runge–Kutta methods ☆

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Abstract

The main purpose of this paper is to discuss the equilibrium attractive properties of a class of multistep Runge–Kutta methods for initial value problems of ordinary differential equations. Some algebraic conditions insuring the equilibrium attractivity are given, and some methods satisfying these algebraic conditions are constructed. Some numerical examples confirm our results.

Keywords: Initial value problems; Ordinary differential equations; Multistep Runge–Kutta methods; Equilibrium attractivity

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1. Introduction

Consider the initial value problems of ordinary differential equations

$$\begin{cases} y'(t) = f(y), & t \geqslant 0, \\ y(0) = y_0 \in R^N, \end{cases} \tag{1.1}$$

where the vector function f: $R^N \to R^N$ satisfies the one-sided Lipschitz condition

$$\langle y - z, f(y) - f(z) \rangle \leqslant \mu \|y - z\|^2 \quad \forall y, z \in R^N, \tag{1.2}$$

$\langle \cdot, \cdot \rangle$ is the standard inner product in R^N with the corresponding norm $\|\cdot, \cdot\|$. This class of the problems (1.1) and (1.2) will be denoted by F_μ. It is well known that two solutions $y(t), z(t)$ of the problem class F_μ with $\mu \leqslant 0$ are contractive, i.e.

$$\|y(t+h) - z(t+h)\| \leqslant e^{\mu h} \|y(t) - z(t)\| \leqslant \|y(t) - z(t)\| \quad \forall t, h \geqslant 0. \tag{1.3}$$

We require that the numerical methods for F_μ ($\mu \leqslant 0$) can reproduce the contractive property (1.3). Thus, some nonlinear stability concepts (such as B-stability, cf. [1–4], (1,0,0)-stability, cf. [1,4–6], etc.) have been introduced, and the corresponding algebraic criteria (i.e. algebraic stability) were established (cf. [3,4,6–8]). But algebraic stability is a strong requirement and is not shared by many other methods (such as the trapezoidal rule, more generally, Lobatto IIIA methods, cf. [3,4,9]) which can be easily implemented. For such methods, it is of interest to look for some simpler properties that still give some insight into the long-time behaviors of numerical methods when applied to the problem class F_0. One such property is the equilibrium attractive property that $\psi(t) := \|f(y(t))\|^2$ ($t \geqslant 0$) is a non-increasing function for any solution $y(t)$. Schmitt and Weiner [10] proved that the problem class F_0 is equilibrium attractive if f is Hölder continuous with exponent $\alpha \in (\frac{1}{2}, 1]$:

$$\|f(u) - f(v)\| \leqslant L \|u - v\|^\alpha \quad \forall u, v \in R^N$$

and discussed the multipoint condition

$$\sum_{0 \leqslant i < k \leqslant s} (-r_{ik}) \langle u_i - v_k, f(u_i) - f(v_k) \rangle \leqslant 0 \quad \forall u_i, v_k \in R^N, \tag{1.4}$$

where $R = (r_{ik})_{i,k=0}^{s}$ is symmetric and nonnegative definite, $e_{s+1}^{\mathrm{T}} = (1, 1, \ldots, 1) \in R^{s+1}$, $Re_{s+1} = 0$. Some authors recently discussed the equilibrium attractive property of linearly implicit ROW-methods and W-methods (cf. [11]), stiffly accurate Runge–Kutta methods (RKMs) (cf. [10]). The other related results can be founded in [12,13, etc.].

Consider the stiffly accurate multistep Runge–Kutta methods (MRKMs) (cf. [4,6,14]) for problems (1.1) and (1.2)

$$Y_i = h\sum_{j=0}^{s} a_{ij}f(Y_j) + \sum_{j=1}^{r}\alpha_j y_{n-1+j}, \quad i = 0, 1, \ldots, s, \tag{1.5a}$$

$$y_{n+r} = h\sum_{j=0}^{s} b_j f(Y_j) + \sum_{j=1}^{r}\alpha_j y_{n-1+j}, \tag{1.5b}$$

where $h > 0$ is the given step-size, a_{ij}, α_k, b_j ($i,j = 0, 1, \ldots, s$; $k = 1, 2, \ldots, r$) are real constants, $\sum_{j=1}^{r}\alpha_j = 1$, $a_{sj} = b_j$, $a_{0j} = 0$, $j = 0, 1, \ldots, s$. The vector Y_i is an approximation $y(t_n + c_i h)$ for $i = 0, 1, \ldots, s$. Here, $t_n = nh$, c_i ($i = 0, 1, \ldots, s$) are real constants and $c_0 = 0$, $c_s = r$, $c_i \neq c_j$ ($i \neq j$). Let

$$A = (a_{ij})_{i,j=0}^{s}, \quad \alpha^{\mathrm{T}} = (\alpha_1, \alpha_2, \ldots, \alpha_r), \quad b^{\mathrm{T}} = (b_0, b_1, \ldots, b_s),$$
$$c^{\mathrm{T}} = (c_0, c_1, \ldots, c_s), \quad C = \mathrm{diag}(c), \quad e_r = (1, 1, \ldots, 1)^{\mathrm{T}} \in R^r$$

and $\tilde{e}_i$ ($i = 1, 2, \ldots, s+1$) are the unit vectors in R^{s+1}. From (1.5), we have

$$Y_0 = \sum_{j=1}^{r}\alpha_j y_{n-1+j}, Y_s = y_{n+r}.$$

In the following sections, the square matrix $W \geqslant 0(> 0)$ denotes W is nonnegative definite (positive definite).

Let us introduce the following simplifying conditions (cf. [1,4,6,14,15])

$$B(\eta) : \alpha^{\mathrm{T}}\chi^k - r^k + kb^{\mathrm{T}}c^{k-1} = 0, \quad k = 1, 2, \ldots, \eta,$$
$$C(\eta) : \tilde{A}\chi^k - c^k + kAc^{k-1} = 0, \quad k = 1, 2, \ldots, \eta,$$

where $\chi = (0, 1, \ldots, r-1)^{\mathrm{T}}$, $\tilde{A} = (\alpha, \alpha, \ldots, \alpha)^{\mathrm{T}}$, and multiplication of vectors is done componentwise.

In this paper, we discuss some equilibrium attractive properties of the method (1.5) for the problem class F_0. Some algebraic conditions insuring the equilibrium attractivity are given, and some methods satisfying these given algebraic conditions are constructed. Some numerical examples confirm our results.

2. Main results and proofs

The stage equation (1.5a) leads to the relation

$$Y_i - Y_k = h\sum_{j=0}^{s}(a_{ij} - a_{kj})f(Y_j), \quad i \neq k, \ 0 \leqslant i, \ k \leqslant s. \tag{2.1}$$

Let

$$f_j = f(Y_j), \quad \Gamma = (\gamma_{ij})_{i,j=0}^{s} = RA + A^{\mathrm{T}}R + \tilde{e}_1\tilde{e}_1^{\mathrm{T}} - \tilde{e}_{s+1}\tilde{e}_{s+1}^{\mathrm{T}},$$

where R are given in (1.4), $\tilde{e}_i$ is the i-th unit vector in R^{S+1}, $i = 1,2,\ldots,S+1$. We can get the following equality:

$$\sum_{0\leqslant i,k\leqslant s}(-r_{ik})\langle Y_i - Y_k, f_i - f_k\rangle = \sum_{i,j=0}^{s}\gamma_{ij}\langle f_i, f_j\rangle + \|f_s\|^2 - \|f_0\|^2 \tag{2.2}$$

by a similar process to the proof process of Theorem 2.1 in [10].

Theorem 2.1. *If there exists a symmetric nonnegative-definite matrix* $R = (r_{ij})_{i,j=0}^{s}$ *such that* $Re_{s+1} = 0$, $\Gamma \geqslant 0$ *and the multipoint condition* (1.4) *holds, then any solution of* (1.5) *satisfies*

$$\|f(Y_s)\| \leqslant \|f(Y_0)\|, \quad i.e.\ \|f(y_{n+r})\| \leqslant \left\| f\left(\sum_{j=1}^{r}\alpha_j y_{n-1+j}\right)\right\|.$$

Proof. The conclusion follows from the assumed conditions and (2.2). □

Theorem 2.1 is an extension of Theorem 2.1 in [10].

Remark 2.1. If the method (1.5) is of B-convergence order $p \geqslant 2$, the vector function f and the solution $y(t)$ of the problem class F_0 satisfies the smooth requirement needed in the following text, then

$$y_{n-1+j} = y(t_{n-1+j}) + O(h^p), \quad j = 1,2,\ldots,r+1.$$

It follows from $\alpha^{T}e_r = 1$ that:

$$Y_0 = y(t_{n-1+r}) + \left(\sum_{j=1}^{r}\alpha_j(j-r)\right)hy'(t_{n-1+r}) + O(h^2).$$

When $B(1)$ holds and $b^{T}e_{s+1} = 1$, we have $\sum_{j=1}^{r}\alpha_j(j-r) = 1 - b^{T}e_{s+1} = 0$. Thus

$$Y_0 = y_{n-1+r} + O(h^2), \quad f(Y_0) = f(y_{n-1+r}) + O(h^2).$$

Therefore, Theorem 2.1 yields

$$\|f(y_{n+r})\| \leqslant \|f(Y_0)\| \leqslant \|f(y_{n-1+r})\| + O(h^2) \leqslant \cdots \leqslant \|f(y_{r-1})\| + O(h).$$

If the initial values $y_{r-1},\ldots,y_1,y_0$ satisfy

$$\|f(y_i)\| \leqslant \|f(y_{i-1})\| + O(h), \quad i = 1,2,\ldots,r-1,$$

then

$$\|f(y_{n+r})\| \leqslant \|f(y_0)\| + O(h).$$

When $f(y_0) = 0$, $\|f(y_{n+r})\| = O(h)$ and $y_{n+r} \to \tilde{y}_0$ for $h \to 0$, here, $f(\tilde{y}_0) = 0$ (cf. [16]).

On the one hand, if $f(\hat{y}) = 0$, $y_i = \hat{y}$ $(i = 0, 1, \ldots, r-1)$, then $y_n = \hat{y}(n \geqslant 0)$, $Y_i = \hat{y}$ $(i = 0, 1, \ldots, s)$ is the solution of (1.5). On the other hand, if $\hat{y}$ is an equilibrium solution of (1.5), i.e., $y_n = \hat{y}$ $(n \geqslant 0)$ and

$$Y_i = h\sum_{j=0}^{s} a_{ij} f(Y_j) + \hat{y}, \quad i = 0, 1, \ldots, s, \tag{2.3a}$$

$$0 = \sum_{j=0}^{s} b_j f(Y_j), \tag{2.3b}$$

then we naturally ask whether $f(\hat{y}) = 0$? This is one of regularity problems of MRKMs, and some results can be found in [17,18]. For (2.3), we have $f(\hat{y}) = 0$ if $\sum_{j=0}^{s} b_j \neq 0$ and $Y_i = \hat{y}$ $(i = 0, 1, \ldots, s)$.

Theorem 2.2. *Assume that there exists a symmetric nonnegative-definite matrix* $R = (r_{ij})_{i,j=0}^{s}$ *such that* $Re_{s+1} = 0$, *and the method* (1.5) *possesses an equilibrium solution.*

(1) *If the multipoint condition* (1.4) *holds, then* $\Gamma \leqslant 0$.
(2) *If the equality of the multipoint condition* (1.4) *holds, then* $\Gamma = 0$.

Proof. Let $\hat{y}$ is an equilibrium solution of (1.5). We have $y_n = \hat{y}(n \geqslant 0)$, $Y_0 = Y_s = \hat{y}$. Thus, the conclusions follows from (2.2) and (2.3). □

Now we give some properties of the simplifying conditions of the method (1.5).

Theorem 2.3. *For the nondegenerate method* (1.5), $C(r-1)$ *cannot hold.*

Proof. If $C(r-1)$ holds, then

$$\tilde{A}\chi^k - c^k + kAc^{k-1} = 0, \quad k = 1, 2, \ldots, r-1. \tag{2.4}$$

Because $c_0 = 0$ and $a_{0j} = 0$ $(j = 0, 1, \ldots, s)$, (2.4) yields $\alpha^{\mathrm{T}}\chi^k = 0$ $(k = 1, 2, \ldots, r-1)$. Moreover, $\alpha_1 = 1, \alpha_2 = \cdots = \alpha_r = 0$. This shows that the method is degenerate. □

In the following sections, we assume that $C(r-2)$ holds. At this time, $\alpha^{\mathrm{T}}\chi^k = 0$ $(k = 1, 2, \ldots, r-2)$, i.e.

$$V_{r-2}(\alpha_2, \ldots, \alpha_{r-1})^{\mathrm{T}} = -\alpha_r(r-1)(1, r-1, \ldots, (r-1)^{r-3})^{\mathrm{T}}, \tag{2.5}$$

where $V_{r-2} = (j^i)_{i,j=1}^{r-2}$ is invertible. From (2.5), we can obtain $\alpha_i = \alpha_i$ (α_r), $i = 2, \ldots, r-1$, here, α_r is a parameter. Moreover, $\alpha_1 = 1 - \sum_{i=2}^{r} \alpha_i$.

Remark 3.2. The method (1.5) with $b_i \neq 0$ $(i = 0, s)$ cannot be algebraically stable (for the nonzero real symmetric matrix $G = (g_{ij})_{i,j=1}^{r} \geqslant 0$ and the nonzero real diagonal matrix $D = \mathrm{diag}(d_0, d_1, \ldots, d_s) \geqslant 0$). In fact, from the definition of the algebraic stability (cf. [1,4,6,15]), we require that the matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \geqslant 0,$$

where

$$M_{22} = A^{\mathrm{T}}D + DA - C_{21}^{\mathrm{T}}GC_{21},$$

$$C_{21} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ b_0 & b_1 & \cdots & b_s \end{pmatrix} \in R^{r\times(s+1)}$$

and

$$(M_{22})_{ij} = -g_{rr}b_ib_j + d_ia_{ij} + d_ja_{ji}, \quad i,j = 0,1,\ldots,s.$$

The formula $M_{22} \geqslant 0$ yields

$$(M_{22})_{00} = -g_{rr}b_0^2 \geqslant 0.$$

Thus, from $b_i \neq 0 (i = 0, s)$, we have

$$g_{rr} = 0, \quad d_ia_{ij} + d_ja_{ji} = 0, \quad i,j = 0,1,2,\ldots,s.$$

Moreover, $d_s = 0$. It follows from a result given in [8], a necessary condition for algebraic stability is

$$De_{s+1} = C_{21}^{\mathrm{T}}Ge_r \quad \text{or} \quad d_i = \kappa b_i, \quad i = 0,1,\ldots,s,$$

where $\kappa = \sum_{j=1}^{r} g_{rj}$. Thus, from $d_s = 0$ and $b_s \neq 0$, we have $\kappa = 0$ and $D = 0$.

3. r-Step RKMs with $s = 1$

When $s = 1$, we have $a_{1j} = b_j$ $(j = 0,1)$, $c_0 = 0$ and $c_1 = r$. Thus, the method (1.5) becomes

$$Y_0 = \sum_{j=1}^{r} \alpha_j y_{n-1+j}, \quad y_{n+r} = Y_1 = h(b_0f(Y_0) + b_1f(Y_1)) + Y_0. \tag{3.1}$$

From $Re_2 = 0$, $R^{\mathrm{T}} = R \geqslant 0$, we have

$$R = a_0\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 - 2a_0b_0 & a_0(b_0 - b_1) \\ a_0(b_0 - b_1) & 2a_0b_1 - 1 \end{pmatrix},$$

where $a_0 = r_{00} > 0$. And the multipoint condition (1.4) holds. In fact,

$$\sum_{0\leqslant i<k\leqslant 1}(-r_{ik})\langle Y_i - Y_k, f(Y_i) - f(Y_k)\rangle = a_0\langle Y_0 - Y_1, f(Y_0) - f(Y_1)\rangle \leqslant 0.$$

It is easy to show that $\Gamma \geqslant 0$ if and only if

$$2a_0b_0 \leqslant 1, \quad 2a_0b_1 \geqslant 1, \quad a_0(b_0 + b_1) = 1. \tag{3.2a}$$

Taking $a_0 = 1$ leads to the conclusion that

$$\Gamma \geqslant 0 \Longleftrightarrow b_0 \leqslant \frac{1}{2}, \quad b_1 \geqslant \frac{1}{2}, \quad b_0 + b_1 = 1. \tag{3.2b}$$

Corollary 3.1. *For $\theta \geqslant \frac{1}{2}$, the θ-scheme*

$$y_{n+1} = y_n + h((1-\theta)f(y_n) + \theta f(y_{n+1})) \tag{3.3}$$

is equilibrium attractive, i.e.

$$\|f(y_{n+1})\| \leqslant \|f(y_n)\|.$$

Proof. Eq. (3.3) can be written as the form of (3.1) with $r = 1$, $c_0 = 0$, $c_1 = 1$ and

$$b_0 = 1 - \theta, \quad b_1 = \theta, \quad \alpha_1 = 1, \quad Y_0 = y_n, \quad Y_1 = y_{n+1}.$$

The conclusion follows from (3.2) and Theorem 2.1. □

Corollary 3.1 has been obtained in [10].

Let $B(r-1)$ hold. We have

$$\alpha^{\mathrm{T}}\chi = r - (b_0 + b_1), \quad \alpha^{\mathrm{T}}\chi^i = r^{i-1}(r - ib_1), \quad i = 2, 3, \ldots, r-1. \tag{3.4}$$

Moreover, (3.4) yields

$$V_{r-1}(\alpha_2, \ldots, \alpha_r)^{\mathrm{T}} = (r - (b_0 + b_1), r(r - 2b_1), \ldots, r^{r-2}(r - (r-1)b_1))^{\mathrm{T}}, \tag{3.5}$$

where $V_{r-1} = (j^i)_{i,j=1}^{r-1}$ is invertible.

When $r = 2$, (3.2a), (3.5) and the equality $\alpha_1 + \alpha_2 = 1$ yield

$$\alpha_2 = 2 - (b_0 + b_1) = 2 - \frac{1}{a_0}, \quad \alpha_1 = b_0 + b_1 - 1 = \frac{1}{a_0} - 1,$$

where $a_0 > 0$ is a free parameter. Moreover, if $a_0 = 1$, then (3.1) degenerates into a one-step RKM.

When $r = 3$, (3.5) yields

$$\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 3 - (b_0 + b_1) \\ 3(3 - 2b_1) \end{pmatrix}. \tag{3.6}$$

And (2.5) (i.e. $C(r-2)$ holds) yields

$$\alpha_2 = -2\alpha_3. \tag{3.7}$$

From (3.2a), (3.6) and (3.7), we have

$$\begin{aligned} &\alpha_2 = -9 + 6b_1, \quad b_0 = 3 - b_1, \quad \alpha_3 = \frac{9}{2} - 3b_1, \\ &\alpha_1 = \frac{11}{2} - 3b_1, \quad a_0 = \frac{1}{3}. \end{aligned} \tag{3.8}$$

Thus, we get a family of equilibrium attractive 3-step RKMs with $s = 1$ and the free parameter $b_1 \geqslant 3/2$. For examples, (3.8) yields by taking $b_1 = 2$

$$\alpha^{\mathrm{T}} = (-1/2, 3, -3/2), \quad b^{\mathrm{T}} = (1, 2), \quad a_0 = 1/3. \tag{3.9}$$

(3.9) satisfies (3.2a), $B(2)$ and $C(1)$.

Applying (3.1) to the model equation

$$y'(t) = \lambda y, \quad Re\lambda \leqslant 0$$

leads to the equality

$$y_{n+r} - \varphi(\bar{h}) \sum_{j=1}^{r} \alpha_j y_{n-1+j} = 0,$$

where

$$\bar{h} = h\lambda, \varphi(\bar{h}) = \frac{1 + \bar{h}b_0}{1 - \bar{h}b_1}.$$

By means of the Schur criteria, we easily prove that the method (3.1) with $r = 2$ is A-stable if

$$0 < \alpha_1, \alpha_2 < 1, \quad b_1 \geqslant b_0, b_0 + b_1 > 0.$$

For examples, we can take

$$a_0 = \frac{2}{3}, \quad \alpha_1 = \alpha_2 = \frac{1}{2}, \quad b_1 = 1, \quad b_0 = \frac{1}{2}. \tag{3.10}$$

(3.10) satisfies (3.2a) and $B(1)$.

4. r-Step RKMs with $s = 2$

When $s = 2$, we have $c = (0, c_1, r)^{\mathrm{T}}$ with $c_1 \neq 0, r$. Thus, the method (1.5) becomes

$$Y_0 = \sum_{j=1}^{r} \alpha_j y_{n-1+j}, \quad Y_1 = h\sum_{j=0}^{2} a_{1j} f(Y_j) + Y_0,$$

$$y_{n+r} = Y_2 = h\sum_{j=0}^{2} b_j f(Y_j) + Y_0. \tag{4.1}$$

For $r = 2$, if $B(3)$ holds, then we have

$$b_1 = \frac{2(\alpha_2 - 2)}{3c_1(c_1 - 2)}, \quad b_2 = 1 - \frac{1}{4}\alpha_2 - \frac{1}{2}b_1 c_1, \tag{4.2a}$$

$$b_0 = 2 - \alpha_2 - b_1 - b_2, \quad \alpha_1 = 1 - \alpha_2, \tag{4.2b}$$

where α_2 and c_1 are free parameters.

For $r = 3$, if $B(3)$ and $C(1)$ hold, then we have

$$b_1 = \frac{9 - 2\alpha_3}{2c_1(3 - c_1)}, \quad b_2 = \frac{3}{2} - \frac{1}{3}\alpha_3 - \frac{1}{3}b_1 c_1, \quad b_0 = 3 - b_1 - b_2, \tag{4.3a}$$

$$\alpha_2 = -2\alpha_3, \quad a_{10} = c_1 - (a_{11} + a_{12}), \quad \alpha_1 = 1 + \alpha_3. \tag{4.3b}$$

For R with $s = 2$, we have

$$r_{00} = r_{11} + r_{22} + 2r_{12}, \quad r_{01} = -r_{11} - r_{12}, \quad r_{02} = -r_{22} - r_{12}.$$

It is easy to prove that $R \geqslant 0$ if and only if

$$r_{11} \geqslant 0, \quad r_{22} \geqslant 0, \quad r_{11}r_{22} \geqslant r_{12}^2, \quad r_{11} + r_{22} \geqslant -2r_{12}. \tag{4.4}$$

We easily show that (1.4) holds if

$$r_{11} \geqslant -r_{12}, \quad r_{22} \geqslant -r_{12}, \quad r_{12} \leqslant 0.$$

Let $r_{11} > 0$, $r_{22} > 0$, $r_{11}r_{22} > r_{12}^2$. For the method (4.1), it follows from $\Gamma = 0$ that

$$a_{11} = -\frac{r_{12}a_{21}}{r_{11}}, \quad a_{12} = -\frac{r_{12}r_{22}}{2(r_{11}r_{22} - r_{12}^2)} - \frac{r_{22}a_{21}}{r_{11}}, \tag{4.5a}$$

$$b_2 = a_{22} = \frac{1 - 2r_{12}a_{12}}{2r_{22}}, \quad a_{10} = -\frac{r_{01}a_{11} + r_{02}a_{21} + r_{12}a_{20}}{r_{11}}, \tag{4.5b}$$

$$b_0 = a_{20} = \frac{r_{12}r_{01}a_{11} + r_{02}r_{12}a_{21} - r_{11}r_{01}a_{12} - r_{11}r_{02}a_{22}}{r_{11}r_{22} - r_{12}^2}, \tag{4.5c}$$

$$r_{12}(r_{22} - 1) = 0, \tag{4.5d}$$

where $a_{21} = b_1$. (4.5) shows that A can be determined by R with $r_{12} = 0$ or $r_{22} = 1$.

When $r_{12} = 0$, (4.5) yields

$$a_{11} = 0, \quad b_2 = a_{22} = \frac{1}{2r_{22}}, \quad a_{12} = -\frac{r_{22}a_{21}}{r_{11}}, \tag{4.6a}$$

$$r_{00} = r_{11} + r_{22}, \quad r_{01} = -r_{11}, \quad r_{02} = -r_{22}. \tag{4.6b}$$

$$a_{10} = \frac{r_{22}a_{21}}{r_{11}}, \quad b_0 = a_{20} = \frac{1}{2r_{22}} - a_{21}, \tag{4.6c}$$

where r_{22}, r_{11} are free parameters. And (1.4) becomes

$$r_{11}\langle Y_0 - Y_1, f(Y_0) - f(Y_1)\rangle + r_{22}\langle Y_0 - Y_2, f(Y_0) - f(Y_2)\rangle \leqslant 0. \tag{4.7}$$

(4.7) holds for $r_{11} > 0$, $r_{22} > 0$.

For $r = 2$ and $r_{12} = 0$, (4.6) and (4.2) lead to

$$\alpha_2 = 2 - \frac{1}{r_{22}}, \quad c_1 = \frac{2(6r_{22} - 5)}{3(2r_{22} - 1)}, \quad r_{22} \neq \frac{1}{2}, \tag{4.8a}$$

$$\alpha_1 = 1 - \alpha_2, \quad a_{21} = b_1 = \frac{2r_{22} - 1}{2r_{22}c_1}, \quad c_1 \neq 0. \tag{4.8b}$$

Therefore, the method (4.1) and the matrix R can be given by (4.6) and (4.8) with the free parameters $r_{11} > 0$ and $r_{22} > 0$. Moreover, if $r_{22} = 1$, then the method (4.1) is degenerate. By taking $r_{11} = 1$ and $r_{22} = 7/8$, (4.6) and (4.8) yield

$$c_1 = 2/9, \quad \alpha^{\mathrm{T}} = (1/7, 6/7), \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 27/16 & 0 & 27/16 \\ -19/14 & 27/14 & 4/7 \end{pmatrix}. \tag{4.9}$$

For $r = 3$ and $r_{12} = 0$, it is easy to show that (4.6) and (4.3) lead to $c_1 = 0$. Thus, the method (4.1) satisfying $B(3)$ and $C(1)$ cannot satisfy $\Gamma = 0$ for $R \geqslant 0$ with $r_{12} = 0$, $r_{11} > 0$, $r_{22} > 0$.

When $r_{22} = 1$, (1.4) holds if

$$r_{11} \geqslant -r_{12}, \quad -1 \leqslant r_{12} \leqslant 0. \tag{4.10}$$

For $r = 2$ and $r_{22} = 1$, (4.2) and (4.5) yield

$$\alpha_2 = \frac{r_{11} - 2r_{12}^2}{r_{11} - r_{12}^2}, \quad \alpha_1 = 1 - \alpha_2,$$

$$\frac{3}{4}(r_{11} - 2r_{12}^2)c_1^2 + \left(3r_{12}^2 - \frac{r_{11}}{2}\right)c_1 + 2r_{12} = 0$$

and b_j, a_{ij} ($i = 1,2$; $j = 0,1,2$) can be given by (4.2) and (4.5) with the free parameters $r_{11} > 0$ and r_{12} satisfying (4.4) and (4.10).

For $r = 3$ and $r_{22} = 1$, (4.3) and (4.5) yield

$$r_{11} = \frac{3}{2}r_{12}^2, \quad c_1 = \frac{5 - 6r_{12}}{r_{12}(3r_{12} + 2)},$$

$$\alpha_3 = \frac{-9(r_{12}c_1 + 2)}{2(r_{12}c_1(2 + c_1) - 2)}, \quad a_{21} = b_1 = \frac{9 - 2\alpha_3}{2c_1(3 - c_1)}$$

and α_j ($j = 1, 2$) and the other a_{ij} ($i = 1, 2$; $j = 0, 1, 2$) can be given by (4.3) and (4.5) with the free parameter $-1 \leqslant r_{12} \leqslant -2/3$.

5. Numerical examples

We choose the method (3.1) with (3.10) and the method (4.1) with (4.9), and apply them to the following two problems, respectively.

Problem 1 (*cf.* [10]).

$$y_1'(t) = -\frac{1}{3}(y_1 - 1)^3 + \frac{20}{3}, \tag{5.1}$$

where $t \geqslant 0$, the initial value $y_1(0) = 3.5$.

Problem 2.

$$y_1'(t) = -\frac{1}{2}y_1 + y_2, \tag{5.2a}$$

$$y_2'(t) = y_1 - 2y_2, \tag{5.2b}$$

where $t \geqslant 0$, the initial values $y_1(0) = -0.4$, $y_2(0) = 0.8$.

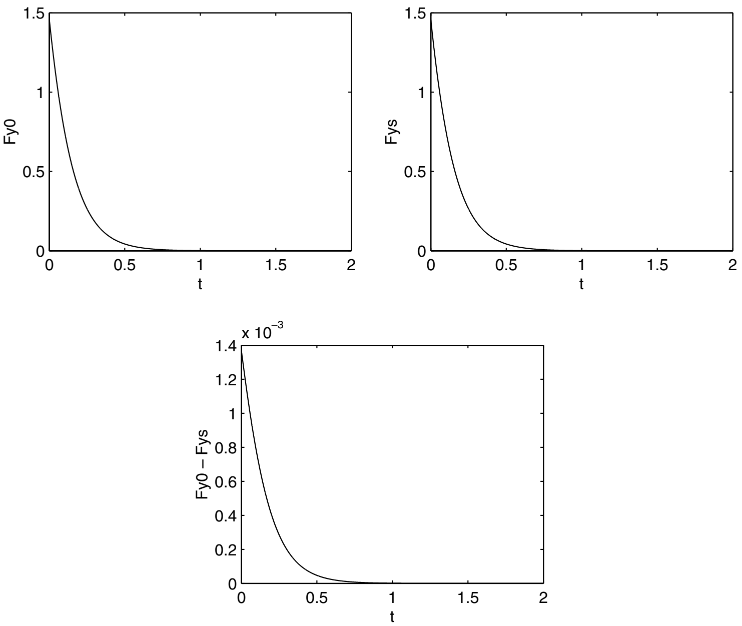

Fig. 1. The method (3.1) with (3.10) for Problem 1.

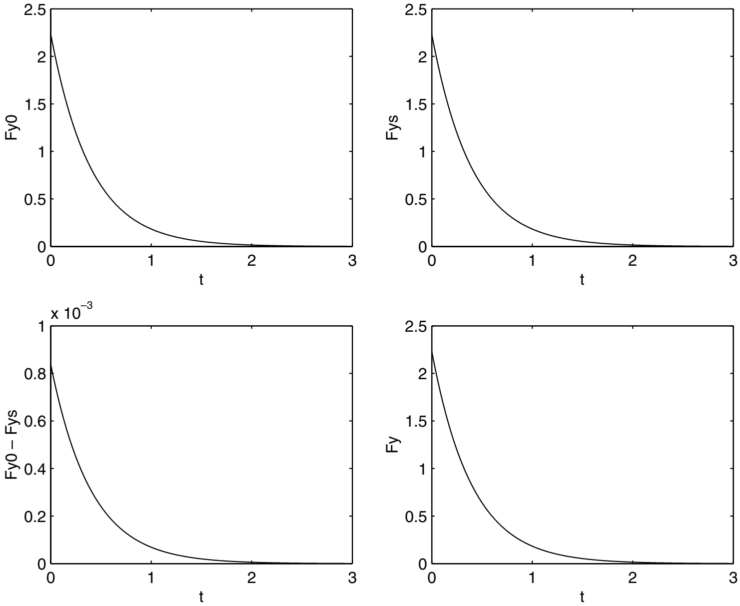

Fig. 2. The method (3.1) with (3.10) for Problem 2.

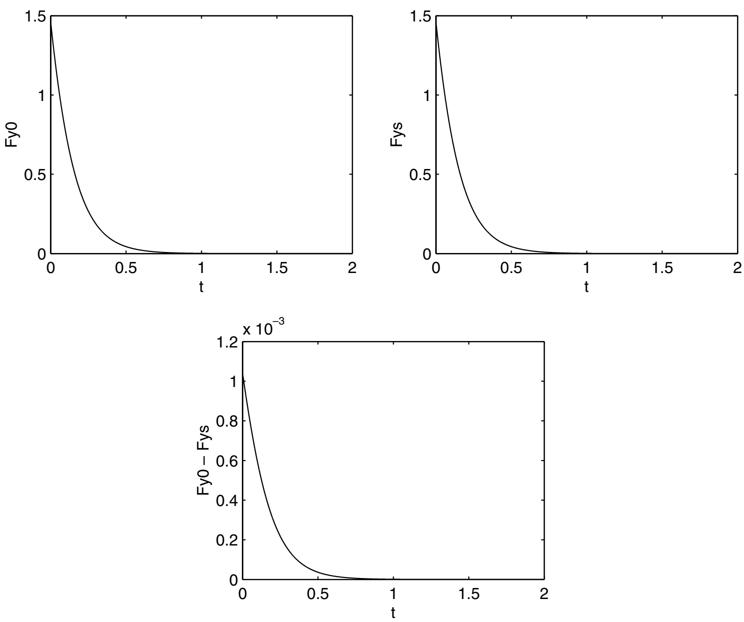

Fig. 3. The method (4.1) with (4.9) for Problem 1.

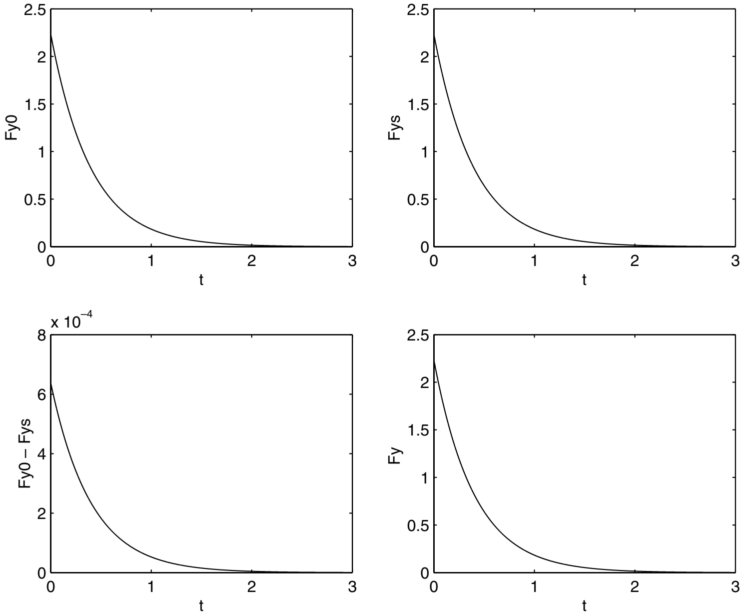

Fig. 4. The method (4.1) with (4.9) for Problem 2.

Problems 1 and 2 belong to F_0, and the above chosen methods satisfy (1.4). Let $Fy0$, Fys and Fy denote $\|f(Y_0)\|$, $\|f(Y_s)\|$ and $\|f(y(t))\|$ with the true solution $y(t)$, respectively. Figs. 1–4 list the numerical results of the above methods with the step-size $h = 0.0001$ and the initial values given by the implicit Euler method for the problems (5.1) and (5.2), and show that the computational results confirm our theoretical results.

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