On algebraic structure and Poisson's theory of mechanico-electrical systems

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Abstract

This letter focuses on studying algebraic structure and the Poisson's theory of mechanico-electrical systems. Based on the relationship between momentum and Hamiltonian, we present Hamilton canonical equations and generalized Hamilton canonical equations and their contravariant algebraic forms for mechanico-electrical systems. The work derives that Lagrange mechanico-electrical systems possess Lie algebraic structure, and then Poisson's theory on holonomic conservative mechanical systems is used by these systems; and Lagrange-Maxwell mechanico-electrical systems possess Lie admitted algebraic structure, and then Poisson's theory on holonomic conservative mechanical systems is halfway used by these systems. Two examples are discussed to illustrate these results.

Key words: algebraic structure, Poisson integral method, mechamico-electrical system
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1. Introduction

It is well known that many methods have been developed to seek invariants of mechanical and physical systems.[1-15] Some important ones Ermakov technique as used by Ray and Reid[1] in a series of papers, Lutzky's approach[2,3] using Noether's theorem, group transformation method of Burgan et al.[4]as used by Ray,[5]dynamical algebraic method of Korsch[6] as used by Kaushal, Korsch and Mishra,[7,8] algebraic structure and Poisson's theory for constrained mechanical systems as used by Mei.[9-14] Among these methods, the invariants of the mechanical systems are studied by using algebraic structure seems to have an additional advantage of having a straightforward extension to the corresponding quantum mechanical systems,[15] and algebraic structure and Poisson's theory are carried out to relativistic Birkhoffian systems by Fu et al.[16-18] These methods while applied successfully to the mechanico-electrical systems have not been explored so far. In this letter, we make an effort in this direction and demonstrate the applications of the algebraic structure and Poisson's theory of dynamical systems to mechanico-electrical systems.

2. Generalized Hamilton canonical equations of mechanico-electrical systems

Mechanico-electrical systems are systems in which a mechanical process and an electromagnetic process are coupled. The mechanical part, constituted by N particles, is described by n generalized coordinates q_l (l=1,…,n), and the electrodynamical part, constituted by m electric circuits, is described by m electric quantities e_k(k=1, …,m). Consider a mechanico-electrical system composed of m circuits consisting of linear conductors and capacitors. Let i_k denote the current, u_k denote the electronic potential, R_k denote the resistance, and C_k denote the capacitance. Then the Lagrangian of the mechanico-electrical

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system is

$$L = T(\boldsymbol{q},\dot{\boldsymbol{q}}) - V(\boldsymbol{q}) - W_e(\boldsymbol{q},\boldsymbol{e}) + W_m(\boldsymbol{q},\boldsymbol{e}) , \tag{1}$$

in this letter, $\boldsymbol{q} = \{q_1, q_2, \cdots, q_n\}, \dot{q} = \{\dot{q}_1, \dot{q}_2, \cdots \dot{q}_n\}, \boldsymbol{e} = \{e_1, e_2. \cdots, e_m\}$ denote generalized coordinate, generalized velocity and generalized electric quantity respectively, $T = \frac{1}{2} a_s(\boldsymbol{q})\dot{q}_s$ and $V = \sum_{s=1}^{n} v_s(\boldsymbol{q})$ are the kinetic energy and the potential energy, and the electric field energy and the magnetic field energy of the *m* th circuit are respectively defined by

$$W_e = \frac{1}{2}\frac{e_k^2}{C_k}, W_m = \frac{1}{2} L_{kr} i_k i_r = \frac{1}{2} L_{kr} \dot{e}_k \dot{e}_r \quad (k, r = 1, \cdots, p) . \tag{2}$$

where $L_{kr}=L_{kr}(\boldsymbol{q})(k\neq r)$ is the mutual-inductance between the *k* th circuit and the *r* th circuit, $L_{kk}=L_{kk}(\boldsymbol{q})$ is the self-inductance of the *k* th return circuit, $C_k=C_k(\boldsymbol{q})$ is the capacitance of *k* th capacitor. We point out that the coupling characteristic of mechanical process and electromagnetic process is expressed in W_e and W_m . For simplicity, we use the convention which the repeated suffixes denote summation in the Letter.

Motion of the system is governed by the Lagrange-Maxwell equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l} + \frac{\partial F}{\partial \dot{q}_l} = Q_l^{'}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{e}_k} - \frac{\partial L}{\partial e_k} + \frac{\partial F}{\partial \dot{e}_k} = u_k \quad (l = 1, \cdots, n; k = 1, \cdots, m) , \tag{3}$$

where $Q_l^{'}$ is non-potential generalized force, *F* is the following dissipative function

$$F = F_e(\dot{\boldsymbol{e}}) + F_m(\boldsymbol{q},\dot{\boldsymbol{q}}) , \tag{4}$$

where $\dot{\boldsymbol{e}} = \{\dot{e}_1, \dot{e}_2, \cdots, \dot{e}_m\} = \boldsymbol{i}$ denotes electric current, the electric dissipative function is given by

$$F_e = \frac{1}{2} R_k i_k^2 = \frac{1}{2} R_k \dot{e}_k^2 \quad (k = 1, \cdots, m) . \tag{5}$$

and F_m is the dissipative function of the viscous frictional damping force.

We now take that the $q_s(s=1,\cdots,n,n+1, \cdots,n+m)$ expresses generalized coordinate which $q_s(s=1,\cdots,n)$ denotes the component of space coordinate, $q_s(s=n+1, \cdots,n+m)$ denotes the component of electrics, then Eq. (3) can be written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial F}{\partial \dot{q}_s} = Q_s \quad (s = 1, \cdots, n, n+1, \cdots, n+m) . \tag{6}$$

where $Q_s(s=1,\cdots,n)$ denotes non-potential generalized force, and $Q_s(s=n+1,\cdots,n+m)$ denotes generalized electrokinetic potential.

The systems are called the Lagrange mechenico-electrical systems satisfying condition $Q_s - \partial F/\partial \dot{q}_s = 0$. Equation (6) yields

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=0 \quad (s=1,\cdots,n,n+1,\cdots,n+m)\,. \tag{7}$$

here the mechanical and electromagnetic process is also coupling each other.

Supposing the generalized momentums of mechenico-electrical systems are

$$p_s=\frac{\partial L}{\partial \dot{q}_s} \quad (s=1,\cdots,n,n+1,\cdots,n+m)\,, \tag{8}$$

and introduce the Hamiltonian of the systems

$$H(t,\boldsymbol{q},\dot{\boldsymbol{q}})=p_s\dot{q}_s-L=p_s\dot{q}_s(t,\boldsymbol{q},\boldsymbol{p})-L(t,q_s,\dot{q}_s(t,\boldsymbol{q},\boldsymbol{p}))=H(t,\boldsymbol{q},\boldsymbol{p})\,, \tag{9}$$

where $\boldsymbol{p}=\{p_1,p_2,\cdots,p_n\}$ denotes generalized momentum. Equation (6) can be shown that

$$\dot{q}_s=\frac{\partial H}{\partial p_s},\quad \dot{p}_s=-\frac{\partial H}{\partial q_s}+(Q_s-\frac{\partial F}{\partial \dot{q}_s})\Big|_{\dot{\boldsymbol{q}}_s=\dot{\boldsymbol{q}}_s(t,\boldsymbol{q},\boldsymbol{p})} \quad (s=1,\cdots,n,n+1,\cdots,n+m), \tag{10}$$

which is called the generalized Hamilton canonical equation of mechanico-electrical systems. Eq. (7) can can be written that

$$\dot{q}_s=\frac{\partial H}{\partial p_s},\quad \dot{p}_s=-\frac{\partial H}{\partial q_s} \quad (s=1,\cdots,n,n+1,\cdots,n+m)\,, \tag{11}$$

which is called the Hamilton canonical equation of Lagrange mechanico-electrical systems. In Eqs. (10) and (11), $\boldsymbol{q}=\{q_1,\cdots,q_n,q_{n+1},\cdots,q_{n+m}\}$,$\boldsymbol{p}=\{p_1,\cdots,p_n,p_{n+1},\cdots,p_{n+m}\}$ denote generalized coordinate and generalized momentum respectively.

3. Contravariant algebraic form of mechanico-electrical systems

Let us introduce contravariant vectors

$$a^\mu=\begin{cases} q^\mu & (\mu=1,\cdots,n,n+1,\cdots,n+m), \\ p_{\mu-(n+m)} & (\mu=n+m+1,\cdots,2(n+m)), \end{cases} \tag{12}$$

Hamiltonian of the system is written in the form

$$H(t,\boldsymbol{q},\boldsymbol{p})=H(t,a^\mu)\,, \tag{13}$$

Using the contravariant tensor

$$(\omega^{\mu\nu})=\begin{pmatrix} 0_{(n+m)\times(n+m)} & I_{(n+m)\times(n+m)} \\ -I_{(n+m)\times(n+m)} & 0_{(n+m)\times(n+m)} \end{pmatrix}, \tag{14}$$

then Hamilton canonical equation (11) is expressed in the contravariant algebraic form

$$\dot{a}^\mu-\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}=0 \quad (\mu,\nu=1,\cdots,n+m,\cdots,n+m,n+m+1,\cdots,2n+2m)\,. \tag{15}$$

For generalized Hamilton canonical equation (10) of Lagrange-Maxwell mechanico-electrical systems, we let

$$(Q_s-\frac{\partial F}{\partial \dot{q}_s})\Big|_{\dot{\boldsymbol{q}}_s=\dot{\boldsymbol{q}}_s(t,\boldsymbol{q},\boldsymbol{p})}=\widetilde{Q}_s=-\Omega_{sk}\frac{\partial H}{\partial p_k} \quad (s,k=1,\cdots,n,n+1,\cdots,n+m)\,, \tag{16}$$

where

$$(\Omega_{sk}) = \begin{pmatrix} \Omega_{11} & 0 & \cdots & 0 \\ 0 & \Omega_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Omega_{(n+m)(n+m)} \end{pmatrix}, \tag{17}$$

then Eq. (10) can be expressed in the contravariant algebraic form.

$$\dot{a}^{\mu} - S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} = 0 \quad (\mu,\nu = 1,\cdots,n,n+1,\cdots,n+m,n+m+1,\cdots,2(n+m)), \tag{18}$$

$$S^{\mu\nu} = \omega^{\mu\nu} + T^{\mu\nu}, \tag{19}$$

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{(n+m)\times(n+m)} & I_{(n+m)\times(n+m)} \\ -I_{(n+m)\times(n+m)} & 0_{(n+m)\times(n+m)} \end{pmatrix}, \tag{20}$$

$$(T^{\mu\nu}) = \begin{pmatrix} 0_{(n+m)\times(n+m)} & 0_{(n+m)\times(n+m)} \\ 0_{(n+m)\times(n+m)} & -\Omega_{kk} \end{pmatrix}. \tag{21}$$

It is obvious that the tensor $S^{\mu\nu}$ composed with anti-symmetry tensor $\omega^{\mu\nu}$ and symmetry tensor $T^{\mu\nu}$.

For example, the Lagrangian of mechanico-electrical system is

$$H = \frac{1}{2}\frac{p_1^2}{m} + \frac{1}{2}\frac{p_2^2}{L_1(x)} + \frac{1}{2}kx^2 - mgx,$$

subjected to non-potential generalized force $Q_1 = c_1 x\dot{x}$, $Q_2 = c_2\dot{e}$, dissipative function $F_e = \frac{1}{2}c_3\dot{e}^2 - c_4\dot{e},\ F_m = c_5 x\dot{x} + c_6\dot{x}^2$. Where $p_1 = m\dot{x},\ p_2 = L_1(x)\dot{e}$ denote generalized momentums, m denotes mass, $k, c_l(l = 1,\cdots,5)$ are constants.

Substituting H into Eq. (16) yields

$$\tilde{Q}_1 = c_1\frac{p_1}{m}x - c_5 x - 2c_6\frac{p_1}{m} = -\Omega_{1k}\frac{\partial H}{\partial p_k}, \quad \tilde{Q}_2 = (c_2 - c_3)\frac{p_2}{L_1(x)} + c_4 = -\Omega_{2k}\frac{\partial H}{\partial p_k},$$

then

$$-\Omega_{11} = c_1 x - c_5\frac{m}{p_1}x + 2c_6, \quad -\Omega_{22} = -(c_3 - c_2) + c_4\frac{L_1(x)}{p_2},$$

$$\left(\omega^{\mu\nu}\right) = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix}, \quad T^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 x - c_5\frac{m}{p_1}x + 2c_6 & 0 \\ 0 & 0 & 0 & -(c_3 - c_2) + c_4\frac{L_1(x)}{p_2} \end{pmatrix}.$$

Substituting $\omega^{\mu\nu}$ and $T^{\mu\nu}$ into Eq. (18) leads to the contravariant algebraic form of the mechanico-electrical system

$$\dot{a}_1 = \frac{p_1}{m}, \quad \dot{a}_2 = \frac{p_2}{L_1(x)}, \quad \dot{a}_3 = \frac{1}{2L_1^2}\frac{\partial L_1}{\partial x}p_2^2 - kx + mg + \left(c_1 x - c_5\frac{m}{p_1}x + 2c_6\right)\frac{p_1}{m},$$
$$\dot{a}_4 = \left[-(c_3 - c_2) + c_4\frac{L_1(x)}{p_2}\right]\frac{p_2}{L_1(x)}.$$

4. Algebraic structure of mechanico-electrical systems

Firstly, we carry out the algebraic structure of Lagrange mechanico-electrical systems

We taking total derivative to a function $A(\boldsymbol{a})$ along the Eq. (15), one has

$$\dot{A} = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\ (\mu,\nu = 1,\cdots,n,n+1,\cdots,n+m,n+m+1,\cdots,2(n+m)), \tag{22}$$

the right-hand side of Eq. (22) is defined as a double-linear product $A \circ H$, i.e.

$$\frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = A \circ H, \tag{23}$$

which satisfies right-hand distributive law, left-hand distributive law and scalar law, so Eq. (23) possesses compatible algebraic structure.

We expanding form (23) yields

$$\frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = \frac{\partial A}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial H}{\partial q^k}, \tag{24}$$

this is the classical Poisson's bracket (A, B), i.e. $(A,B) = A \circ B$. It is well known that Poisson's bracket satisfies anti-symmetry

$$A \circ B + B \circ A = 0, \tag{25}$$

and Jacobi identical equation

$$A \circ (B \circ C) + B \circ (C \circ A) + C \circ (A \circ B) = 0, \tag{26}$$

equations (25) and (26) are also called Lie algebra axiom, then one has

Theorem 1 Equations of motion of Lagrange mechanico-electrical systems possess the compatible algebraic structure as well as the Lie algebraic structure.

Secondly, one presents the algebraic structure of Lagrange-Maxwell mechanico-electrical systems.

We taking total derivative to a function $A(\boldsymbol{a})$ along Eq. (18), and this derivative is defined as a product

$$\dot{A}(\boldsymbol{a}) = \frac{\partial A}{\partial a^\mu}S^{\mu\nu}\frac{\partial H}{\partial a^\nu} \underset{=\!=}{def}\ A \circ H, \tag{27}$$

this product also satisfies right-hand distributive law, left-hand distributive law and scalar law, and then the Eq. (18) possesses the compatible algebraic structure.

If Eq. (18) possesses the Lie algebraic structure, the Eq. (27) satisfies Eqs. (25) and (26). When considering Eqs. (19)~(21), the Eqs. (25) and (26) lead to conditions with respect to $T^{\mu\nu}$

$$T^{\mu\nu} + T^{\nu\mu} = 0, \tag{28}$$

$$T^{\tau\rho}\frac{\partial T^{\mu\nu}}{\partial a^{\rho}}+T^{\mu\rho}\frac{\partial T^{\nu\tau}}{\partial a^{\rho}}+T^{\nu\rho}\frac{\partial T^{\tau\mu}}{\partial a^{\rho}}=0$$
$$(\mu,\nu,\tau,\rho=1,\cdots,n,n+1,\cdots,n+m,n+m+1,\cdots,2n+2m), \tag{29}$$

Further, we carry out

$$\Omega_{kk}=0,\quad (k=1,\cdots,n,n+1,\cdots,n+m), \tag{30}$$

$$Q_s=0\quad (s=1,\cdots,n,n+1,\cdots,n+m). \tag{31}$$

Therefore, Eq.（18）have no Lie algebraic structure, we have

Theorem 2 The equations of motion of Lagrange-Maxwell mechanic-electrical systems have not Lie algebraic structure.

We can prove that the Lagrange-Maxwell mechanico-electrical systems possess the Lie admitted algebraic structure. For Eq. (27), we define a new product

$$[A,B]\underline{\underline{def}}\ A\circ B-B\circ A, \tag{32}$$

it satisfies anti-symmetry

$$[A,B]+[B,A]=0, \tag{33}$$

and Jacobi identical equation

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0, \tag{34}$$

Expanding Eq.（34），the $S^{\mu\nu}$ satisfies the following form

$$(S^{\mu\rho}-S^{\rho\mu})\frac{\partial}{\partial a^{\rho}}(S^{\nu\tau}-S^{\tau\nu})+(S^{\nu\rho}-S^{\rho\nu})\frac{\partial}{\partial a^{\rho}}(S^{\tau\mu}-S^{\mu\tau})+$$
$$(S^{\tau\rho}-S^{\rho\tau})\frac{\partial}{\partial a^{\rho}}(S^{\mu\nu}-S^{\nu\mu})=0 \tag{36}$$
$$(\mu,\nu,\tau,\rho=1,\cdots,n,n+1,\cdots,n+m,n+m+1,\cdots,2(n+m)).$$

Using Eq. (19) ~Eq.（21），equation (36) leads to

$$(T^{\mu\rho}-T^{\rho\mu})\frac{\partial}{\partial a^{\rho}}(T^{\nu\tau}-T^{\tau\nu})+(T^{\nu\rho}-T^{\rho\nu})\frac{\partial}{\partial a^{\rho}}(T^{\tau\mu}-T^{\mu\tau})$$
$$+(T^{\tau\rho}-T^{\rho\tau})\frac{\partial}{\partial a^{\rho}}(T^{\mu\nu}-T^{\nu\mu})=0. \tag{37}$$

$T^{\mu\nu}$ is a symmetry tensor, then Eq (37) is evident. The new product (32) possesses Lie algebraic structure, then Eq. (27) possesses Lie admitted algebraic structure. We give

Theorem 3 Equations of motion of Lagrange-Maxwell mechanico-electrical systems possess the compatible algebraic structure as well as Lie admitted algebraic structure.

5. Poisson's theory of mechanico-electrical systems

It is well known that there are two main contents in the Poisson's integration theory of conservative holonomic systems. One is the Poisson conditions on the first integral. The other is the Poisson theorem. This theory points out that the Poisson's bracket of two first integrals, which are not involution, is also a first integral. We now give generalizations of the Poisson's theory in mechanico-electrical systems.

5.1 Poisson's theory of Lagrange mechanico-electrical systems

We have known that the theoretical foundation of Possion integral method is equation of motion of systems possessing Lie algebraic structure.[19] Lagrange mechanico-electrical system（11）possesses Lie algebraic structure, then the Poisson integral methods of conservative holonomic dynamical systems can all be used in the systems. Then we have

Proposition 1 The necessary and sufficient condition on which $I(a^{\mu},t)$ $(\mu=1,\cdots,n+m)$ is a first integral of the Lagrange mechanico-electrical system (11) is that the $I(a^{\mu},t)$ satisfies

$$\frac{\partial I}{\partial t}+(I,H)=0\,, \tag{38}$$

where

$$(I,H)=\frac{\partial I}{\partial q_k}\frac{\partial H}{\partial p_k}-\frac{\partial I}{\partial p_k}\frac{\partial H}{\partial q_k}\quad (k=1,\cdots,n,n+1,\cdots,n+m)\,, \tag{39}$$

is a Poisson bracket.

Proof In fact, the I is a first intefral if and only if $\dot{I}=0$. Notice that

$$\frac{dI}{dt}=\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^{\mu}}\Omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}=\frac{\partial I}{\partial t}+(I,H)=0\,.$$

Equation (39) is called the Poisson condition of the first integral for Lagrange mechanico-electrical systems.

Proposition 2 $H=c$ is a first integral of Lagrange mechanico-electrical system, if the Hamiltonian of the system does not depend evidently on time t.

Proposition 3 For the Lagrange mechanico-electrical system (11), which possessing two first integrals $I_1(a^{\mu},t)$ and $I_2(a^{\mu},t)$ have not involution, then theirs Poisson bracket（I_1, I_2）is also a first integral of the system.

Proof : Supposing the Lagrange mechanico-electrical system (11) possesses the following two first integrals having not involution

$$I_1(a^{\mu},t)=c_1,\quad I_2(a^{\mu},t)=c_2\,, \tag{40}$$

which satisfy Poisson conditions

$$\frac{\partial I_1}{\partial t}+(I_1,H)=0,\quad \frac{\partial I_2}{\partial t}+(I_2,H)=0\,. \tag{41}$$

We take the following operation

$$\frac{\partial}{\partial t}(I_1,I_2)=(\frac{\partial I_1}{\partial t},I_2)+(I_1,\frac{\partial I_2}{\partial t})+\frac{\partial I_1}{\partial a^{\mu}}\frac{\partial \omega^{\mu\nu}}{\partial t}\frac{\partial I_2}{\partial t}\,, \tag{42}$$

in respect that

$$\frac{\partial \omega^{\mu\nu}}{\partial t}=0\,,$$

then one has

$$\frac{\partial}{\partial t}(I_1,I_2)=(\frac{\partial I_1}{\partial t},I_2)+(\frac{\partial I_2}{\partial t},I_1)\,, \tag{43}$$

using Lie algebraic axioms（25），（26）lead to

$$((I_1,I_2),H)=-((I_2,H),I_1)-((H,I_1),I_2)=(I_1,(I_2,H))+((I_1,H),I_2)\,, \tag{44}$$

combining Eqs. (43) and（44）and considering Eq. (41) yield

$$\frac{\partial}{\partial t}(I_1,I_2)+((I_1,I_2),H)=((\frac{\partial I_1}{\partial t})+(I_1,H),I_2)+(I_1,\frac{\partial I_2}{\partial t}+(I_2,H))=0\,. \tag{45}$$

then（I_1,I_2）being also a first integral of the system is obtained in Eq. (45).

Proposition 4 For the Lagrange mechanico-electrical system（11）,which possesses a first integral $I(a^\mu,t)$ containing t , and Hamiltonian does not depend evidently on t, then $\frac{\partial I}{\partial t},\frac{\partial^2 I}{\partial t^2},\cdots$, are also first integrals of the system.

Proof: By partial differentiating（38）with respect to t , we obtain

$$\frac{\partial}{\partial t}\frac{\partial I}{\partial t}+(\frac{\partial I}{\partial t},H)+(I,\frac{\partial H}{\partial t})=0\,,$$

where H does not depend evidently on t, one has

$$\frac{\partial}{\partial t}\frac{\partial I}{\partial t}+(\frac{\partial I}{\partial t},H)=0\,. \tag{46}$$

then $\frac{\partial I}{\partial t}$ is a first integral of Eq (11). Similar, one can prove that the $\frac{\partial^2 I}{\partial t^2},\frac{\partial^3 I}{\partial t^3},\cdots$, are also the first integrals of the system.

Proposition 5 For the Lagrange mechanico-electrical system (11), which possesses a first integral $I(a^\mu,t)$ containing a^ρ, and Hamiltonian H does not depend evidently on a^ρ, then $\frac{\partial I}{\partial a^\rho},\frac{\partial^2 I}{\partial a^{\rho^2}},\cdots$, are also first integrals of the system.

Proof: By partial differentiating (38) with respect to a^ρ, we have

$$\frac{\partial}{\partial a^\rho}\frac{\partial I}{\partial t}+(\frac{\partial I}{\partial a^\rho},H)+(I,\frac{\partial H}{\partial a^\rho})=0\,,$$

where H does not depend evidently on a^ρ, one has

$$\frac{\partial}{\partial t}\frac{\partial I}{\partial a^\rho}+(\frac{\partial I}{\partial a^\rho},H)=0\,. \tag{47}$$

then $\frac{\partial I}{\partial a^\rho}$ is a first integral of the Eq. (11). Similar we can prove $\frac{\partial^2 I}{\partial a^{\rho^2}},\frac{\partial^3 I}{\partial a^{\rho^3}},\cdots$, are also first integrals of the system.

5.2 Poisson's theory of Lagrange-Maxwell mechanico-electrical systems

Poisson's theory still point out that[19] the part of Poisson integral methods may be used by such systems, when the equation of motion of the systems possesses Lie admitted algebraic structure, and the

Poisson theorem are not hold in the systems.

Proposition 6 The necessary and sufficient condition on which $I(a^{\mu},t)$ $(\mu=1,\cdots,n+m)$ is a first integral of the Lagrange-maxwell mechanico-electrical system (18) is that the $I(a^{\mu},t)$ satisfies

$$\frac{\partial I}{\partial t}+[I,H]=0\,, \tag{48}$$

where

$$[I,H]=I\circ H=\frac{\partial I}{\partial a^{\mu}}S^{\mu\nu}\frac{\partial I}{\partial a^{\nu}} \quad (\mu,\nu=1,\cdots,n,n+1,\cdots,n+m,n+m+1,\cdots,2n+2m).$$

is Poisson bracket, and $S^{\mu\nu}$ is given by Eqs. (19)-(21).

Proposition 7 *H*=*C* is a first integral of the conservative and no dissipated Lagrange-Maxwell mechanico-electrical system (18), if the Hamiltonian *H* of the system does not depend evidently on time *t*.

Proof: Substituting *I*=*H* into Eq. (48) leads to

$$\frac{\partial H}{\partial t}+[H,H]=\frac{\partial H}{\partial t}+\frac{\partial H}{\partial a^{\mu}}S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}=\frac{\partial H}{\partial t}+\frac{\partial H}{\partial a^{\mu}}T^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}=\frac{\partial H}{\partial t}-\Omega_{kk}(\frac{\partial H}{\partial a^{n+m+s}})^{2}, \tag{49}$$

using Eq. (15), we obtain

$$\frac{\partial H}{\partial t}+[H,H]=\frac{\partial H}{\partial t}+\widetilde{Q}_{s}(\frac{\partial H}{\partial a^{n+m+s}}) \quad (s=1,\cdots,n,n+1,\cdots,n+m). \tag{50}$$

then *H*=*C* is first integral of Lagrange-Maxwell mechanico-electrical system (18) for the case of $\widetilde{Q}_{s}=0$ and no dissipation, when *H* does not depend evidently on *t*.

Proposition 8 For the Lagrange-Maxwell mechanico-electrical system（18）, which possesses a first integral $I(a^{\mu},t)$ containing *t* ,when Hamiltonian *H* , non-potential generalized force Q_s and the dissipative function *F* do not depend evidently on *t*, $\frac{\partial I}{\partial t},\frac{\partial^{2} I}{\partial t^{2}},\cdots$, are also first integrals of the system.

Proof: By partial differentiating (48) with respect to *t*, we can obtain

$$\frac{\partial}{\partial t}(\frac{\partial I}{\partial t})+[\frac{\partial I}{\partial t},H]+\frac{\partial\widetilde{Q}_{s}}{\partial t}\frac{\partial I}{\partial a^{\mu}}+\frac{\partial I}{\partial a^{\mu}}S^{\mu\nu}\frac{\partial}{\partial a^{\nu}}(\frac{\partial H}{\partial t})=0, \tag{51}$$

when Hamiltonian *H*, non-potential generalized force Q_s and the dissipative function *F* do not depend evidently on *t*, i. e. $\frac{\partial}{\partial t}\widetilde{Q}_{s}$=0, $\frac{\partial H}{\partial t}=0$, one has

$$\frac{\partial}{\partial t}(\frac{\partial I}{\partial t})+[\frac{\partial I}{\partial t},H]=0\,. \tag{52}$$

namely $\frac{\partial I}{\partial t}$ is first integral of the system (18). Similar, we can prove $\frac{\partial^{2} I}{\partial t^{2}},\cdots$, are also first integrals of the system.

Proposition 9 For the Lagrange-Maxwell mechanico-electrical system (18), which possesses a first integral $I(a^{\mu},t)$ containing a^{ρ}, and Hamiltonian *H* , non-potential generalized force Q_s and the dissipative function F do not depend evidently on a^ρ, then $\frac{\partial I}{\partial a^\rho}, \frac{\partial^2 I}{\partial a^{\rho^2}}, \cdots$, are also first integrals of the system.

Proof: By partial differentiating (48) with respect to a^ρ, we can lead to

$$\frac{\partial}{\partial t}(\frac{\partial I}{\partial a^\rho}) + \frac{\partial}{\partial a^\mu}(\frac{\partial I}{\partial a^\rho})S^{\mu\nu}\frac{\partial H}{\partial a^\nu} + \frac{\partial \widetilde{Q}_s}{\partial a^\rho}\frac{\partial I}{\partial a^{n+m+s}} + \frac{\partial I}{\partial a^\mu}S^{\mu\nu}\frac{\partial}{\partial a^\nu}(\frac{\partial H}{\partial a^\rho}) = 0. \tag{53}$$

when Hamiltonian H, non-potential general force Q_s and the dissipative function F do not depend evidently on a^ρ, i.e. $\frac{\partial}{\partial a^\rho}\widetilde{Q}_s$ =0, $\frac{\partial H}{\partial a^\rho} = 0$, we have

$$\frac{\partial}{\partial t}(\frac{\partial I}{\partial a^\rho}) + \frac{\partial}{\partial a^\mu}(\frac{\partial I}{\partial a^\rho})S^{\mu\nu}\frac{\partial H}{\partial a^\nu} = \frac{\partial}{\partial t}(\frac{\partial I}{\partial a^\rho}) + [\frac{\partial I}{\partial a^\rho}, H] = 0\,. \tag{54}$$

namely $\frac{\partial I}{\partial a^\rho}$ is first integral of the system (18), Similar, we can prove $\frac{\partial^2 I}{\partial a^{\rho 2}}, \cdots$, are also first integrals of the system.

6. An example

Consider a circuitry of dynamoelectric transducer, which memorizes mechanical vibration. Let m denote the mass of armature, k denote the stiffness factor, $L_1=L_1(x)=bx$ (b is constant) denote the self-inductance of winding. The circuitry constituted by winding, battery and resistance, where x denotes the displacement of plumb line, which is calculated from originally length of spring. E denotes the electromotive force of battery, and R denotes the resistance. We choose that displacement x and electric quantity q are expressed by general coordinates. We, further, suppose that the armature subjected nonconservative force $E\dot{q}/\dot{x} + R\dot{q}^2/\dot{x}$. Let us study algebraic structure and Poisson integral approach.

The kinetic energy and magnetic energy of the system is

$$T = \frac{1}{2}m\dot{x} + \frac{1}{2}bx\dot{q}^2, \tag{55}$$

potential energy is

$$V = \frac{1}{2}kx^2 - mgx, \tag{56}$$

and dissipative function is

$$F = -\frac{1}{2}R\dot{q}^2, \tag{57}$$

Lagrange function of the system is written in the form

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}bx\dot{q}^2 - \frac{1}{2}kx^2 + mgx\,. \tag{58}$$

Taking the general momentums and velocities of system are

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \dot{x} = \frac{p_1}{m}, \\ p_2 &= \frac{\partial L}{\partial \dot{q}} = bx\dot{q}, \quad \dot{q} = \frac{p_2}{bx}, \end{aligned} \tag{59}$$

then Hamiltonian of the system is expressed by

$$H=\frac{1}{2}\frac{p_1^2}{m}+\frac{1}{2}\frac{p_2^2}{bx}+\frac{1}{2}kx^2-mgx\,. \tag{60}$$

Equations of motion of the system are described by part canonical form

$$\begin{aligned}&\dot{x}=\frac{p_1}{m},\quad \dot{q}=\frac{p_2}{bx},\\ &\dot{p}_1=-\frac{\partial H}{\partial x}-E\frac{\dot{q}}{\dot{x}}+R\frac{\dot{q}^2}{\dot{x}}=-kx+mg+\frac{1}{2b}\frac{p_2^2}{x^2}-E\frac{\dot{q}}{\dot{x}}+R\frac{\dot{q}^2}{\dot{x}},\\ &\dot{p}_2=-\frac{\partial H}{\partial q}+E-R\dot{q}=E-R\dot{q},\end{aligned} \tag{61}$$

in which

$$\widetilde{Q}_1=-E\frac{\dot{q}}{\dot{x}}+R\frac{\dot{q}^2}{\dot{x}}=-E\frac{mp_2}{bxp_1}+R\frac{mp_2^2}{b^2x^2},\quad \widetilde{Q}_2=E-R\frac{p_2}{bx}\,. \tag{62}$$

Let

$$a_1=x,a_2=q,a_3=p_1,a_4=p_2\,, \tag{63}$$

Equation（59）is written as contravariant algebraic form，one has

$$\dot{a}^u-S^{\mu\nu}\frac{\partial H}{\partial a^\nu}=0\,,\qquad (\mu,\nu=1,\cdots,4) \tag{64}$$

where

$$S^{\mu\nu}=\omega^{\mu\nu}+T^{\mu\nu}\,,$$

$$(\omega^{\mu\nu})=\begin{bmatrix}0_{2\times2} & I_{2\times2}\\ -I_{2\times2} & 0_{2\times2}\end{bmatrix},\ (T^{\mu\nu})=\begin{bmatrix}0_{2\times2} & 0_{2\times2}\\ 0_{2\times2} & -\Omega_{kk}\end{bmatrix}.$$

Using Eqs. (15) and (62) , we obtain

$$-\Omega_{11}=-E\frac{m^2a_4}{ba_1a_3^2}+R\frac{m^2a_4^2}{b^2a_1^2a_3^2},\quad -\Omega_{22}=E\frac{ba_1}{a_4}-R\,. \tag{65}$$

Therefore, we give

$$\dot{a}_1=\frac{a_3}{m},\quad \dot{a}_2=\frac{a_4}{ba_1},\quad \dot{a}_3=ka_1-mg+\frac{1}{2b}\frac{a_4^2}{a_1{}^2}-E\frac{ma_4}{ba_1a_3}+R\frac{ma_4^2}{b^2a_1^2a_3},\quad \dot{a}_4=E-R\frac{a_4}{ba_1}, \tag{66}$$

when

$$\frac{\partial H}{\partial t}=0,\quad \widetilde{Q}_s\dot{q}_s=\widetilde{Q}_1\dot{x}+\widetilde{Q}_2\dot{q}=-E\frac{a_4}{ba_1}+R\frac{a_4^2}{b^2a_1^2}+E\frac{a_4}{ba_1}-R\frac{a_4{}^2}{b^2a_1{}^2}=0\,. \tag{67}$$

we know, from the proposition (7), that Hamiltonian of the system

$$H=\frac{1}{2}\frac{a_3^2}{m}+\frac{1}{2}\frac{a_4^2}{ba_1}+\frac{1}{2}ka_1^2-mga_1=C_1\,, \tag{68}$$

is first integral. Using proposition 6, we can obtain

$$\frac{\partial I}{\partial t}+I\circ H=\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^{\mu}}\omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}+\frac{\partial I}{\partial a^{\mu}}T^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}=\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^{1}}\frac{a_3}{m}+\frac{\partial I}{\partial a_2}\frac{a_4}{ba^{1}}+$$
$$\frac{\partial I}{\partial a^{3}}(-E\frac{ma_4}{ba_1a_3}+R\frac{ma_4^2}{b^2a_1^2a_3}+\frac{1}{2}\frac{a_4^2}{ba_1^2}-ka_1+mg)+\frac{\partial I}{\partial a_4}(E-R\frac{a_4}{ba_1})=0, \tag{69}$$

Equation (69) is one-order linear homogeneous partial differential equation, its characteristic equation is

$$\frac{dt}{1}=\frac{mda_1}{a_3}=\frac{ba_1da_2}{a_4}=\frac{a_3da_3}{(-E\frac{ma_4}{ba_1}+\frac{Rma_4^2}{b^2a_1^2}+\frac{a_3a_4^2}{2ba_1^2}-ka_1a_3+mga_3)}=\frac{da_4}{E-\frac{Ra_4}{ba_1}}, \tag{70}$$

We, from Eq. (70), can obtain the following integrals

$$I_1=ma_1-\int a_3dt=c_1, \tag{71}$$

$$I_2=a_2-\int\frac{a_4}{ba_1}dt=c_2, \tag{72}$$

$$I_3=\frac{1}{2}a_3^2+E\frac{ma_2}{ba_1}+\frac{mRa_2a_4}{b^2a_1^2}+\frac{a_2a_3a_4}{2ba_1^2}-\frac{1}{2}ka_1^2+mga_1=c_3, \tag{73}$$

$$I_4=a_2+\frac{a_4}{R}+\frac{Eba_1}{R^2}\ln(E-\frac{Ra_4}{ba_1})=C_4. \tag{74}$$

the first integral I_3 includes a_2, using proposition 9, from I_3 leads to the new integral

$$I_5=E\frac{m}{ba_1}+\frac{mRa_4}{b^2a_1^2}+\frac{a_3a_4}{2ba_1^2}=c_5. \tag{75}$$

7. Conclusion

The algebraic structure and the Poisson's theory are extended to mechanico-electrical systems in this letter. The results here present significant approaches to seek conserved quantities in mechanico-electrical systems.

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