Non-overlapping Domain Decomposition Methods with A New Class of Multilevel Solvers[1]

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Abstract

In this paper we are concerned with non-overlapping domain decomposition methods for the second-order elliptic problems in three-dimensional domains. In this method, the interface variable is chosen as Dirichlet boundary data. We first propose a new kind of multilevel preconditioner for the interface operator associated with this domain decomposition. Then, we apply the multilevel idea to developing a substructuring preconditioner with inexact solvers. For the new multilevel preconditioners, the computational complexity may be optimal. It will be shown that the condition number of the preconditioned system grows only as the logarithm of the dimension of the local problem associated with an individual substructure, and is independent of possible jumps of the coefficient in the elliptic equation.

Key words. domain decomposition, interface space, multilevel decomposition, preconditioner, condition number, harmonic extension, inexact solver

AMS(MOS) subject classification 65F10, 65N30, 65N55

1 Introduction

Non-overlapping domain decomposition methods (DDMs) have been shown to be powerful techniques for solving large-scale partial differential equations, especially for solving the partial differential equations with large jump coefficients. How to construct an efficient preconditioner for the resulting interface operators is a core problem in the study of non-overlapping DDMs. The construction of this preconditioner has been investigated from various ways and to various models in literature, see, for example, [8], [12], [18], [20], [24], [27], [28], [31], [35], [36] and [40]. This kind of preconditioner consists of two ingredients: one coarse solver and a set of local solvers. It seems difficult to design cheap local solvers (for general quasi-uniform grids), which can be viewed as preconditioners for Steklov-Poincare operators (or Poincare-Steklov operators). To avoid explicit derivation of the interface operators, one can use substructuring methods associated with Dirichlet interface variable [5], or consider saddle-point problems associated with Neumann interface variable (Lagrange multiplier)[21]. Another problem in non-overlapping DDMs is how to develop an efficient substructuring method with inexact solvers (to internal variable)[4], [15].

In the present paper, we propose a new kind of multilevel method for preconditioning the interface operator. The main idea is to make a multilevel domain decomposition to the interface, instead of multilevel grid decomposition to the triangulation on the interface [32] and [34]. The main difference between the new method and the traditional multilevel method is that a series of refined grids is unnecessary for the new method. Also the new method is different from the multipole method [29] (or the $\mathcal{H}$ matrix method [14]). The former is a preconditioning technique, and can be implemented in parallel. But, the latter is a complete approximate technique, and is not a natural parallel algorithm. As applications

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of the new multilevel method, we develop new substructuring methods with inexact solvers. In particular, we design a substructuring method with two-level domain decomposition. The new substructuring methods depend on multilevel discrete harmonic extensions, which correspond to the new multilevel decomposition mentioned above. It will be shown that all the new methods has almost the optimal convergence. Namely, the condition number of the preconditioned system grows only as the logarithm of the dimension of the local problem associated with an individual substructure, and is independent of possible jumps of the coefficient in the elliptic equation.

The outline of the remainder of the paper is as follows. In §2, we describe some notations, and give some technical lemmas. In §3, we introduce two basic decompositions for the interface space. The multilevel decomposition for the interface space is presented in §4. In §5, we construct a multilevel preconditioner for the interface operator. In §6, we discuss a variant of BPS substructuring method, and investigate what is the problem of substructuring methods with inexact solvers. The new substructuring method with inexact solvers is developed in §7. In §8, we propose a kind of two-level substructuring method. In §9, we construct approximate harmonic extensions, which are involved in §7 and §8.

2 Preliminaries

2.1 Domain decomposition

Let Ω be a bounded polyhedron in $\mathcal{R}^3$. Consider the model problem

$$\begin{cases} -div(\omega\nabla u) = f, \quad in \ \ \Omega, \\ \quad u = 0, \quad on \ \ \partial\Omega, \end{cases} \tag{2.1}$$

where $\omega \in L^\infty(\Omega)$ is a positive function.

Let $H_0^1(\Omega)$ denote the standard Sobolev space, and let $(\cdot,\cdot)$ denote the $L^2(\Omega)$-inner product. The weak formulation of (2.1) in $H_0^1(\Omega)$ is then given by the following.

Find $u \in H_0^1(\Omega)$ such that

$$\mathcal{A}(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$, and

$$\mathcal{A}(u,v) = \int_\Omega \omega\nabla u \cdot \nabla v dp.$$

We will apply a kind of non-overlapping domain decomposition method to solving (2.2). For simplicity of exposition, we consider only the case with matching grids in this paper.

Let $\mathcal{T}_h = \{\tau_i\}$ be a regular and quasi-uniform triangulation of Ω with $\tau_i's$ being non-overlapping simplexes of size h ($\in (0,1]$). The set of notes of $\mathcal{T}_h$ is denoted by $\mathcal{N}_h$. We then define $V_h(\Omega)$ to be the piecewise linear finite element subspace of $H_0^1(\Omega)$ associated with $\mathcal{T}_h$:

$$V_h(\Omega) = \{v \in H_0^1(\Omega): \ v|_\tau \in \mathcal{P}_1 \ \ \forall \tau \in \mathcal{T}_h\},$$

where $\mathcal{P}_1$ is the space of linear polynomials. Then the finite element approximation for (2.2) is to find $u_h \in V_h(\Omega)$ such that

$$\mathcal{A}(u_h,v_h) = (f,v_h), \quad \forall v_h \in V_h(\Omega). \tag{2.3}$$

Let Ω be decomposed into the union of N polyhedrons $\Omega_1,\cdots,\Omega_{\rm N}$, which satisfy $\Omega_i \cap \Omega_j = \emptyset$ when $i \neq j$. We assume that each $\partial\Omega_k$ can be written as a union of boundaries of elements in $\mathcal{T}_h$, and all Ω_k are of size H in the usual sense (see [5] and [40]). Without loss of generality, we assume that the coefficient $\omega(p)$ is piecewise constant, then each subdomain Ω_k is chosen such that $\omega(p)$ equals to a constant ω_k in Ω_k. Note that $\{\Omega_k\}$ may not constitute a triangulation of Ω. Set

$$V_h(\Omega_k) = \{v|_{\Omega_k} : \ \forall v \in V_h\}.$$

If Ω_i and Ω_j are two neighboring subdomains, we denote by Γ_{ij} the common face of Ω_i and Ω_j (i.e., $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$). The union of all Γ_{ij} is denoted by Γ, which is called the interface. In this paper, we choose Dirichlet data as the interface unknown.

2.2 Some notations

• (local) interface space

As usual, we define the (global) interface space by

$$W_h(\Gamma) = \{v|_\Gamma : \ \forall v \in V_h\}.$$

In the following, a subset G of Γ are always understood as an open set. The closure of G is denoted by $\bar{G}$. For a subset G of Γ, define

$$W_h(G) = \{v|_G : \ \forall v \in V_h\},$$

and

$$W_h(\bar{G}) = \{v|_{\bar{G}} : \ \forall v \in V_h\}.$$

• interpolation operators and integration average

For a subset G of Γ, define the interpolation operator $I_G^0 : W_h(\Gamma) \to W_h(\Gamma)$ as

$$(I_G^0\phi_h)(p) = \begin{cases} \phi_h(p), & \text{if } p \in \mathcal{N}_h \cap G, \\ 0, & \text{if } p \in \mathcal{N}_h \cap (\Gamma\backslash G). \end{cases}$$

Similarly, we define $I_{\bar{G}}^0 : W_h(\Gamma) \to W_h(\Gamma)$ by

$$(I_{\bar{G}}^0\phi_h)(p) = \begin{cases} \phi_h(p), & \text{if } p \in \mathcal{N}_h \cap \bar{G}, \\ 0, & \text{if } p \in \mathcal{N}_h \cap (\Gamma\backslash \bar{G}). \end{cases}$$

Define

$$\tilde{W}_h(G) = \{\tilde{\phi}_h \in W_h(\Gamma) : \ \tilde{\phi}_h = I_G^0\phi_h \text{ for some } \phi_h \in W_h(\Gamma)\}.$$

For a function $\varphi_h \in W_h(\Gamma)$, let $\gamma_G(\varphi_h)$ denote the integration average of φ_h on G.

• sets of vertices, edges, faces and subdomains

For convergence, let $\mathcal{F}_\Gamma$ denote the set of the faces Γ_{ij}. Besides, let $\mathcal{V}_\Gamma$ and $\mathcal{E}_\Gamma$ denote the set of the vertices and the set of coarse edges associated with the decomposition

$$\bar{\Omega} = \bigcup \bar{\Omega}_k,$$

respectively.

Set

$$\mathcal{Q}_{ij} = \{k : \ \partial\Omega_k \cap \partial\Gamma_{ij} \neq \emptyset\},$$

and

$$\Omega_{ij} = \Omega_i \bigcup \Omega_j, \quad \Gamma_{ij} \in \mathcal{F}_\Gamma.$$

For an edge $\text{E} \in \mathcal{E}_\Gamma$, define

$$\mathcal{Q}_\text{E} = \{k : \ \text{E} \subset \partial\Omega_k\},$$

and

$$\Omega_\text{E} = \bigcup_{k\in\mathcal{Q}_\text{E}} \Omega_k, \quad \text{E} \in \mathcal{E}_\Gamma.$$

• inner-product and scaling norm

For a subset G of Γ, let $\langle\cdot,\cdot\rangle_G$ denote the L^2 inner product on G. In particular, the $\langle\cdot,\cdot\rangle_\Gamma$ is abbreviated as $\langle\cdot,\cdot\rangle$. Let $\|\cdot\|_{0,\,G}$ denote the norm induced from $\langle\cdot,\ \cdot\rangle_G$.

For a sub-faces G of Γ, let H_G denote the "size" of G. Define the scaling norm

$$\|\varphi\|_{\frac{1}{2},\,G} = (|\varphi|^2_{\frac{1}{2},\,G} + H_G^{-1}\|\varphi\|^2_{0,\,G})^{\frac{1}{2}}, \quad \forall\varphi \in H^{\frac{1}{2}}(G).$$

As usual, let $\|\cdot\|_{-\frac{1}{2},\,G}$ denote the dual norm of $\|\cdot\|_{\frac{1}{2},\,G}$.

For convenience, set

$$\langle\varphi,\psi\rangle_{*,\partial\Omega_k} = \int_{\partial\Omega_k}\int_{\partial\Omega_k} \frac{(\varphi(p)-\varphi(q))(\psi(p)-\psi(q))}{|p-q|^3} ds(p)ds(q), \quad \varphi,\ \psi \in W_h(\Gamma),$$

and

$$\langle\varphi,\psi\rangle_{*,\Gamma} = \sum_{k=1}^{\text{N}} \omega_k\langle\varphi,\psi\rangle_{*,\partial\Omega_k}, \quad \varphi,\ \psi \in W_h(\Gamma).$$

It is clear that $\langle\cdot,\cdot\rangle_{*,\Gamma}$ is an inner-product on $W_h(\Gamma)$. The norm reduced by $\langle\cdot,\cdot\rangle_{*,\Gamma}$ is denoted by $\|\cdot\|_{*,\Gamma}$.

• spectrally equivalences

For simplicity, we will frequently use the notations $\lesssim$ and $\eqsim$. For any two non-negative quantities x and y, $x \lesssim y$ means that $x \leq Cy$ for some constant C independent of mesh size h, subdomain size d and the related parameters. Similarly, $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

2.3 Motivation

Let $\varphi_h \in W_h(\Gamma)$ denote the Dirichlet interface unknown. After eliminating the internal variable, one gets the interface equation (see [33] for the details)

$$S_h\varphi_h = g_h. \tag{2.4}$$

The core problem in non-overlapping DDMs (including substructuring methods) is how to construct an efficient preconditioner for the interface operator S_h. The standard manner for such construction is to decompose the interface Γ into

$$\Gamma = \bigcup_{r=1}^{\text{M}} \Gamma_r,$$

where Γ_r is a local interface Γ_{ij} or a boundary $\partial\Omega_k$. For each Γ_r, let $W_r(\Gamma) \subset W_h(\Gamma)$ be a local space determined by $W_h(\Gamma_r)$ $(r = 1, \cdots, \text{M})$. Besides, choose $W_0(\Gamma)$ as a (small and global) coarse subspace of $W_h(\Gamma)$. Consider the space decomposition for $W_h(\Gamma)$

$$W_h(\Gamma) = W_0(\Gamma) + \sum_{r=1}^{\text{M}} W_r(\Gamma). \tag{2.5}$$

Then, a preconditioner for S_h can be defined based on the decomposition (2.5), provided that suitable solvers S_r on $W_r(\Gamma)$ $(r = 0, \cdots, \text{M})$ are designed. Three typical ideas for such construction were advanced in [5], [8] and [27], and various variants of these ideas were proposed in the last twenty years (see [20], [31], [33], [35] and [40] for the detailed introduction).

It is clear that, when the coarse subspace $W_0(\Gamma)$ has low dimensions, each local subspace $W_r(\Gamma)$ $(r = 1, \cdots, \text{M})$ would possess high dimensions. This means that implementation of the action of each local solver S_r^{-1} is in general very expensive, since S_r $(r = 1, \cdots, \text{M})$ resulting in a dense stiffness matrix (with a relatively high order). It is known that how to preconditioning S_k is a difficult topic for a general quasi-uniform triangulation $\mathcal{T}_h$ (refer to [39]). In the following, we propose a new preconditioned technique for S_r based on a multilevel decomposition for Γ_r. As a example, we choose each Γ_r to be a local interface Γ_{ij}. Here, we use only multilevel *domain* decomposition for Γ_{ij}, instead of multilevel *grid* decomposition for $\mathcal{T}_{ij}$. Because of this, the traditional refined grids are unnecessary. With the multilevel decomposition for Γ_{ij}, a new multilevel space decomposition for $W_h(\Gamma)$ is got. This space decomposition consists of multilevel (local) coarser subspaces and many small local subspaces. All the coarse subspaces and local subspaces have low dimensions.

2.4 Technical tools

By the definition of $\|\cdot\|_{*,\,\Gamma}$, we have

$$\langle S_h\varphi_h, \varphi_h\rangle \eqsim \sum_{k=1}^{\text{N}} \omega_k |\varphi_h|^2_{\frac{1}{2},\partial\Omega_k} = \|\varphi_h\|^2_{*,\Gamma}, \quad \forall \varphi_h \in W_h(\Gamma). \tag{2.6}$$

After designing a suitable space decomposition for $W_h(\Gamma)$, one needs to show that the decomposition is stable under the norm $\|\cdot\|_{*,\Gamma}$. For this purpose, we need to use a class of edge lemmas and face lemmas.

The following results can be found in [40].

Lemma 2.1 *Let* E *and* F *denote an edge and a face of* Ω_k*, respectively. Then,*

$$\|I^0_{\text{E}}\varphi_h\|_{\frac{1}{2},\partial\Omega_k} \lesssim [1 + \log(H/h)]^{\frac{1}{2}} \|\varphi_h\|_{\frac{1}{2},\partial\Omega_k}, \quad \forall \varphi_h \in W_h(\partial\Omega_k), \tag{2.7}$$

and

$$\|\varphi_h\|_{\frac{1}{2},\partial\text{F}} \lesssim [1 + \log(H/h)]^{\frac{1}{2}} \|\varphi_h\|_{\frac{1}{2},\partial\Omega_k}, \quad \forall \varphi_h \in W_h(\partial\Omega_k). \tag{2.8}$$

□

The following inequality can be proved as in Lemma 6.2 in [22], together with the standard technique.

Lemma 2.2 *Let* E *be an edge of* Ω_k*. Then,*

$$\|\varphi_h - \gamma_{\text{E}}(\varphi_h)\|_{\frac{1}{2},\partial\Omega_k} \lesssim [1 + \log(H/h)]^{\frac{1}{2}} |\varphi_h|_{\frac{1}{2},\partial\Omega_k}, \quad \forall \varphi_h \in W_h(\partial\Omega_k). \tag{2.9}$$

□

Let F denote a face Γ_{ij} itself or a sub-face of Γ_{ij}. We assume that F is just the union of some elements τ on the face Γ_{ij}, and possesses the size d in the usual sense. Here, we do not require that F is a polygon with finite edges.

Lemma 2.3 *Let* F *be defined above. Then,*

$$|I^0_{\text{F}}\varphi_h|_{H^{\frac{1}{2}}_{00}(\text{F})} \lesssim [1 + \log(d/h)] \|\varphi_h\|_{\frac{1}{2},\text{F}}, \quad \forall \varphi_h \in W_h(\text{F}). \tag{2.10}$$

Proof. Since F may be a nonstandard polygon, one can not get this result in a direct way. Our idea is to map F into a rectangle on the plane of F.

Choose four nodes p_1, p_2, p_3 and p_4 on ∂F in order. Assume that $|p_1p_2|$, $|p_2p_3|$, $|p_3p_4|$ and $|p_4p_1|$ has almost the same length. Let ℓ denote the straight line through p_1 and p_3, and let ℓ' denote the straight line through p_2 and p_4. The unique intersection of ℓ and ℓ' is denoted by o, which is almost the barycenter of F by the assumption. Make a sufficiently small rectangle D containing F, so that ℓ and ℓ' are just two diagonal lines of D. Moreover, we require that o is the center of D. It is clear that the size of D is also d. For each node $p \in \bar{\text{F}}$ ($p \neq o$), draw a line ℓ_p through o and p. Let p' and p'' denote the intersection of ℓ_p with ∂F and with ∂D, respectively. Define the mapping F by

$$F(p) = \frac{|op''|}{|op'|}p, \quad \forall p \in \mathcal{N}_h \cap \bar{\text{F}} \ (p \neq o).$$

If o is just a node, then define $F(o) = o$. It is easy to see that F maps ∂F onto ∂D, and maps the meshes on F onto a quasi-uniform and regular meshes on D. Moreover, the four vertices of D equal to $F(p_i)$ ($i = 1, 2, 3, 4$), which are just four nodes of such resulting triangulation $\mathcal{T}_h^D$ (with the diameter h).

Let $W_h(\bar{D})$ denote the linear finite element space associated with $\mathcal{T}_h^D$. For $\varphi_h \in W_h(\bar{\text{F}})$, define $F\varphi_h \in W_h(\bar{D})$ by

$$(F\varphi_h)(F(p)) = \varphi_h(p), \quad \forall p \in \mathcal{N}_h \cap \text{F}.$$

By the discrete norm, one can verify that

$$|I^0_{\text{F}}\varphi_h|_{H^{\frac{1}{2}}_{00}(\text{F})} \eqsim |I^0_D(F\varphi_h)|_{H^{\frac{1}{2}}_{00}(D)}. \tag{2.11}$$

It is known that (see [5] and [40])

$$|I^0_D(F\varphi_h)|_{H^{\frac{1}{2}}_{00}(D)} \lesssim [1 + \log(d/h)]\|F\varphi_h\|_{\frac{1}{2},D}.$$

Plugging this in (2.11), and using the discrete norm again, gives (2.10).

□

3 Basic space decompositions for $W_h(\Gamma)$

The new multilevel decomposition for $W_h(\Gamma)$ will depend on a basic (one-level) decomposition for $W_h(\Gamma)$. Different basic decompositions can resulting in different multilevel decompositions, although the multilevel decompositions has the same idea. For simplicity of exposition, we first in this section introduce one-level space decompositions for $W_h(\Gamma)$. There are various ways to design this kind of decomposition (refer to [5], [8] and [40]). How to design such decomposition is not the main interest of this paper. In the following we only consider two examples of decompositions, which are different from the existing methods.

3.1 Decomposition (I)

The first decomposition depends on a set of restriction operators, which satisfy the partition of unit on $W_h(\Gamma)$.

For a face Γ_{ij}, define the local space on $\partial\Gamma_{ij}$

$$W_h(\partial\Gamma_{ij}) = \{\tilde{\varphi}_h \in W_h(\Gamma) : \ \tilde{\varphi}_h = I^0_{\partial\Gamma_{ij}}\varphi_h \text{ for some } \varphi_h \in W_h(\Gamma)\}.$$

For a subdomain Ω_k and a node $p \in \mathcal{N}_h$, let $n_k(p)$ denote the number of faces (of Ω_k) containing p. For example, if p is an internal node in a face of Ω_k, then $n_k = 1$; if p is an internal node in an edge of Ω_k, then $n_k = 2$. Define the extension operator $\theta_{ij} : W_h(\Gamma) \rightarrow \tilde{W}_h(\bar{\Gamma}_{ij})$ ($\subset W_h(\Gamma)$) as

$$(\theta_{ij}\phi_h)(p) = \begin{cases} \frac{\sqrt{\omega_i}+\sqrt{\omega_j}}{\sum\limits_{k\in\mathcal{Q}_{ij}} n_k(p)\sqrt{\omega_k}}\phi_h(p), & if\ p \in \mathcal{N}_h \cap \bar{\Gamma}_{ij}, \\ 0, & if\ p \in \mathcal{N}_h \cap (\Gamma \backslash \bar{\Gamma}_{ij}), \end{cases} \quad (\phi_h \in W_h(\Gamma)).$$

It can be verified that

(i) each θ_{ij} satisfies

$$\theta_{ij}\phi_h(p) = \phi_h(p), \quad \forall p \in \mathcal{N}_h \cap \Gamma_{ij} \ \text{ and } \ |\theta_{ij}\phi_h(p)| \le |\phi_h(p)|, \quad \forall p \in \Gamma; \tag{3.1}$$

(ii) all the operators θ_{ij} satisfy the partition of unit

$$\sum_{\Gamma_{ij}\subset\Gamma} \theta_{ij} = \textsc{i} \quad \text{on } W_h(\Gamma). \tag{3.2}$$

Hereafter, $\textsc{i}$ denotes the identity operator on $W_h(\Gamma)$.

Define

$$\hat{W}_h^0(\Gamma) = span\{\theta_{ij}1 : \text{ for each face } \Gamma_{ij} \subset \Gamma\}.$$

It is easy to see that

$$W_h(\Gamma) = \hat{W}_h^0(\Gamma) + \sum_{\Gamma_{ij}} W_h(\partial\Gamma_{ij}) + \sum_{\Gamma_{ij}} \tilde{W}_h(\Gamma_{ij}).$$

Theorem 3.1 *For any $\phi_h \in W_h(\Gamma)$, there exists a decomposition*

$$\phi_h = \hat{\phi}_0 + \sum_{\Gamma_{ij}} \phi_{ij}^{\partial} + \sum_{\Gamma_{ij}} \phi_{ij}, \quad \text{with } \phi_0 \in W_h^0(\Gamma),\ \phi_{ij}^{\partial} \in W_h(\partial\Gamma_{ij}) \ \text{ and } \ \phi_{ij} \in \tilde{W}_h(\Gamma_{ij}),$$

such that

$$\|\hat{\phi}_0\|_{*,\Gamma}^2 + \sum_{\Gamma_{ij}} \|\phi_{ij}^{\partial}\|_{*,\Gamma}^2 + \sum_{\Gamma_{ij}} \|\phi_{ij}\|_{*,\Gamma}^2 \lesssim [1 + \log(H/h)]^2 \|\phi_h\|_{*,\Gamma}^2. \tag{3.3}$$

Proof. For convenience, set $\psi_{ij} = \varphi_h - \gamma_{\Gamma_{ij}}(\phi_h)$. We first define

$$\hat{\phi}_0 = \sum_{\Gamma_{ij}} \theta_{ij}\gamma_{\Gamma_{ij}}(\phi_h), \text{ and } \phi_{ij} = I_{\Gamma_{ij}}^0 \psi_{ij}.$$

Then, we define $\phi_{ij}^{\partial} \in W_h(\Gamma)$ by

$$\phi_{ij}^{\partial}(p) = \begin{cases} (\theta_{ij}\psi_{ij})(p), & if\ p \in \mathcal{N}_h \cap \partial\Gamma_{ij}, \\ 0, & if\ p \in \mathcal{N}_h \cap (\Gamma \backslash \partial\Gamma_{ij}). \end{cases}$$

It is clear that

$$\phi_0 \in W_h^0(\Gamma),\ \phi_{ij}^{\partial} \in W_h(\partial\Gamma_{ij}) \ \text{ and } \ \phi_{ij} \in \tilde{W}_h(\Gamma_{ij}).$$

Moreover, we have by (3.1) and (3.2)

$$\phi_h = \hat{\phi}_0 + \sum_{\Gamma_{ij}} \phi_{ij}^{\partial} + \sum_{\Gamma_{ij}} \phi_{ij}. \tag{3.4}$$

As usual, we get by (2.10) with $\mathrm{F} = \Gamma_{ij}$

$$\sum_{\Gamma_{ij}} \|\phi_{ij}\|_{*,\Gamma}^2 \lesssim [1+\log(H/h)]^2 \|\phi_h\|_{*,\Gamma}^2. \tag{3.5}$$

It is clear that

$$\|\phi_{ij}^{\partial}\|_{*,\Gamma}^2 = \sum_{k\in\mathcal{Q}_{ij}} \omega_k |\phi_{ij}^{\partial}|_{\frac{1}{2},\partial\Omega_k}^2. \tag{3.6}$$

By the definition of ϕ_{ij}^{∂} , we have (note (3.1)

$$|\phi_{ij}^{\partial}|_{\frac{1}{2},\partial\Omega_i}^2, \ |\phi_{ij}^{\partial}|_{\frac{1}{2},\partial\Omega_j}^2 \lesssim \|\theta_{ij}\psi_{ij}\|_{0,\partial\Gamma_{ij}}^2 \lesssim \|\psi_{ij}\|_{0,\partial\Gamma_{ij}}^2. \tag{3.7}$$

Let $k \in \mathcal{Q}_{ij}$ but $k \neq i, j$. Then, $\partial\Gamma_{ij} \cap \partial\Omega_k$ is an (closed) edge, which is denoted by $\bar{\mathrm{E}}_{ij,k}$. Since $(\phi_{ij}|_{\partial\Omega_k})(p)$ vanishes except at $p \in \mathcal{N}_h \cap \bar{\mathrm{E}}_{ij,k}$, we get

$$|\phi_{ij}^{\partial}|_{\frac{1}{2},\partial\Omega_k}^2 \lesssim \|\theta_{ij}\psi_{ij}\|_{0,\mathrm{E}_{ij,k}}^2 \lesssim (\frac{\sqrt{\omega_i}+\sqrt{\omega_j}}{\sqrt{\omega_i}+\sqrt{\omega_j}+\sqrt{\omega_k}})^2 \|\psi_{ij}\|_{0,\mathrm{E}_{ij,k}}^2 \quad (k\in\mathcal{Q}_{ij}, \ r\neq i,j). \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), yields

$$\|\phi_{ij}^{\partial}\|_{*,\Gamma}^2 \lesssim (\omega_i+\omega_j)\|\psi_{ij}\|_{0,\partial\Gamma_{ij}}^2. \tag{3.9}$$

Here, we have used the inequality

$$\omega_k(\frac{\sqrt{\omega_i}+\sqrt{\omega_j}}{\sqrt{\omega_i}+\sqrt{\omega_j}+\sqrt{\omega_k}})^2 \leq 2(\omega_i+\omega_j).$$

Using (3.9), together with (2.8), yields

$$\|\phi_{ij}^{\partial}\|_{*,\Gamma}^2 \lesssim (\omega_i+\omega_j)[1+\log(H/h)]^2\|\psi_{ij}\|_{\frac{1}{2},\Gamma_{ij}}^2.$$

We further get by Poincare inequality

$$\|\phi_{ij}^{\partial}\|_{*,\Gamma}^2 \lesssim [1+\log(H/h)]^2(\omega_i|\phi_h|_{\frac{1}{2},\partial\Omega_i}^2 + \omega_j|\phi_h|_{\frac{1}{2},\partial\Omega_j}^2).$$

Thus,

$$\sum_{\Gamma_{ij}} \|\phi_{ij}^{\partial}\|_{*,\Gamma}^2 \lesssim [1+\log(H/h)]^2 \sum_{k=1}^{\mathrm{N}} \omega_k|\phi_h|_{\frac{1}{2},\partial\Omega_k}^2 = [1+\log(H/h)]^2\|\phi_h\|_{*,\Gamma}^2. \tag{3.10}$$

It is easy to see that $\langle\phi_{ij}^{\partial}, \phi_{rl}^{\partial}\rangle_{*,\Gamma} = 0$ when $\partial\Gamma_{ij} \cap \partial\Gamma_{rl} = \emptyset$. It follows by (3.4) that

$$\begin{aligned} \|\hat{\phi}_0\|_{*,\Gamma}^2 &\lesssim \|\phi_h\|_{*,\Gamma}^2 + \|\sum_{\Gamma_{ij}}\phi_{ij}^{\partial}\|_{*,\Gamma}^2 + \|\sum_{\Gamma_{ij}}\phi_{ij}\|_{*,\Gamma}^2 \\ &\lesssim \|\phi_h\|_{*,\Gamma}^2 + \sum_{\Gamma_{ij}}\|\phi_{ij}^{\partial}\|_{*,\Gamma}^2 + \sum_{\Gamma_{ij}}\|\phi_{ij}\|_{*,\Gamma}^2. \end{aligned}$$

This, together with (3.10) and (3.6), leads to

$$\|\hat{\phi}_0\|_{*,\Gamma}^2 \lesssim [1+\log(H/h)]^2\|\phi_h\|_{*,\Gamma}^2. \tag{3.11}$$

By (3.10), (3.6) and the above inequality, gives (3.3).

□

3.2 Decomposition (II)

The second decomposition is based on an edge-face decomposition for Γ.

Define

$$W_h^0(\Gamma) = \{\phi_h \in W_h(\Gamma): \ \phi_h(p) = C_{\mathrm{E}}, \forall p \in \mathcal{N}_h \cap \mathrm{E} \text{ for } \mathrm{E} \in \mathcal{E}_\Gamma; \\ \phi_h(p) = C_{\mathrm{F}}, \forall p \in \mathcal{N}_h \cap \mathrm{F} \text{ for } \mathrm{F} \in \mathcal{F}_\Gamma\}.$$

Hereafter, C_G denotes a constant depending on G.

It is easy to see that

$$W_h(\Gamma) = W_h^0(\Gamma) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \tilde{W}_h(\mathrm{E}) + \sum_{\Gamma_{ij}} \tilde{W}_h(\Gamma_{ij}).$$

Theorem 3.2 *For any $\phi_h \in W_h(\Gamma)$, there exists a decomposition*

$$\phi_h = \phi_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \phi_{\mathrm{E}} + \sum_{\Gamma_{ij}} \phi_{ij} \tag{3.12}$$

with

$$\phi_0 \in W_h^0(\Gamma); \ \phi_{\mathrm{E}} \in \tilde{W}_h(\mathrm{E}) \ \ \textit{and} \ \ \phi_{ij} \in \tilde{W}_h(\Gamma_{ij}),$$

such that

$$\|\phi_0\|_{*,\Gamma}^2 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \|\phi_{\mathrm{E}}\|_{*,\Gamma}^2 + \sum_{\Gamma_{ij}} \|\phi_{ij}\|_{*,\Gamma}^2 \leq [1+\log(H/h)]^2 \|\phi_h\|_{*,\Gamma}^2. \tag{3.13}$$

Proof. For $\mathrm{E} \in \mathcal{E}_\Gamma$ and a face Γ_{ij}, set

$$\phi_{\mathrm{E}} = I_{\mathrm{E}}^0[\phi_h - \gamma_{\mathrm{E}}(\phi_h)], \ \text{and} \ \phi_{ij} = I_{\Gamma_{ij}}^0[\varphi_h - \gamma_{\Gamma_{ij}}(\phi_h)].$$

It is clear that $\phi_{\mathrm{E}} \in \tilde{W}_h(\mathrm{E})$ and $\phi_{ij} \in \tilde{W}_h(\Gamma_{ij})$. Define $\phi_0 \in W_h^0(\Gamma)$ by

$$\phi_0(p) = \begin{cases} \phi_h(p), & if \ p \in \mathcal{N}_h \cap \mathcal{V}_\Gamma, \\ \gamma_{\mathrm{E}}, & if \ p \in \mathcal{N}_h \cap \mathrm{E} \ (\mathrm{E} \in \mathcal{E}_\Gamma), \\ \gamma_{\mathrm{F}}, & if \ p \in \mathcal{N}_h \cap \mathrm{F} \ (\mathrm{F} \in \mathcal{F}_\Gamma). \end{cases}$$

It is easy to see that

$$\phi_h = \phi_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \phi_{\mathrm{E}} + \sum_{\Gamma_{ij}} \phi_{ij}. \tag{3.14}$$

Then, we have by (2.7)

$$\begin{aligned} \|\phi_{\mathrm{E}}\|_{*,\Gamma}^2 &= \sum_{k\in\mathcal{Q}_{\mathrm{E}}} \omega_k |\phi_{\mathrm{E}}|_{\frac{1}{2},\partial\Omega_k}^2 \\ &\lesssim [1+\log(H/h)] \sum_{k\in\mathcal{Q}_{\mathrm{E}}} \omega_k \|\phi_h - \gamma_{\mathrm{E}}(\phi_h)\|_{\frac{1}{2},\partial\Omega_k}^2. \end{aligned}$$

This, together with (2.9), yields

$$\begin{aligned} \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \|\phi_{\mathrm{E}}\|_{*,\Gamma}^2 &\lesssim [1+\log(H/h)]^2 \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \sum_{k\in\mathcal{Q}_{\mathrm{E}}} \omega_k |\phi_h|_{\frac{1}{2},\partial\Omega_k}^2 \\ &\lesssim [1+\log(H/h)]^2 \sum_{k=1}^{\mathrm{N}} \omega_k |\phi_h|_{\frac{1}{2},\partial\Omega_k}^2 \\ &= [1+\log(H/h)]^2 \|\phi_h\|_{*,\Gamma}^2. \end{aligned} \tag{3.15}$$

On the other hand, one get by (2.10) (with $\mathrm{F} = \Gamma_{ij}$) and Poincare inequality

$$\begin{aligned} |\phi_{ij}|_{\frac{1}{2},\partial\Omega_i} &\lesssim [1+\log(H/h)]\|\phi_h - \gamma_{\Gamma_{ij}}(\phi_h)\|_{\frac{1}{2},\partial\Omega_i} \\ &\lesssim [1+\log(H/h)]|\phi_h|_{\frac{1}{2},\partial\Omega_i}. \end{aligned} \tag{3.16}$$

Similarly, we have

$$|\phi_{ij}|_{\frac{1}{2},\Omega_j} \lesssim [1+\log(H/h)]|\phi_h|_{\frac{1}{2},\partial\Omega_j}. \tag{3.17}$$

Combining (3.16) and (3.17), leads to

$$\|\phi_{ij}\|^2_{*,\Gamma} \lesssim [1+\log(H/h)]^2[\omega_i|\phi_h|^2_{\frac{1}{2},\partial\Omega_i} + \omega_j|\phi_h|^2_{\frac{1}{2},\partial\Omega_j}].$$

Thus,

$$\sum_{\Gamma_{ij}} \|\phi_{ij}\|^2_{*,\Gamma} \lesssim [1+\log(H/h)]^2 \sum_{k=1}^{\mathrm{N}} \omega_k|\phi_h|^2_{\frac{1}{2},\partial\Omega_k} = \|\phi_h\|^2_{*,\Gamma}. \tag{3.18}$$

It follows by (3.14) that

$$\begin{aligned} \|\phi_0\|^2_{*,\Gamma} &\lesssim \|\phi_h\|^2_{*,\Gamma} + \|\sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \phi_{\mathrm{E}}\|^2_{*,\Gamma} + \|\sum_{\Gamma_{ij}} \phi_{ij}\|^2_{*,\Gamma} \\ &\lesssim \|\phi_h\|^2_{*,\Gamma} + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \|\phi_{\mathrm{E}}\|^2_{*,\Gamma} + \sum_{\Gamma_{ij}} \|\phi_{ij}\|^2_{*,\Gamma}. \end{aligned}$$

This, together with (3.15) and (3.18), leads to

$$\|\phi_0\|^2_{*,\Gamma} \lesssim [1+\log(H/h)]^2\|\phi_h\|^2_{*,\Gamma}. \tag{3.19}$$

Now, the inequality (3.13) is a direct consequence of (3.15), (3.18) and the above inequality. □

Remark 3.1 *The decomposition (I) and decomposition (II) have their respectively merits. On the one hand, the coarse subspace $\hat{W}^0_h(\Gamma)$ in the decomposition (I) has smaller dimensions than the coarse subspace $W^0_h(\Gamma)$ in the decomposition (II). On the other hand, the basis of $W^0_h(\Gamma)$ is easier to get than the basis of $\hat{W}^0_h(\Gamma)$, which involves the extensions θ_{ij}.*

Remark 3.2 *The decompositions proposed in this section are different from the standard decompositions (see subsection 2.3). If merging $\sum_{\Gamma_{ij}} W_h(\partial\Gamma_{ij})$ into $\hat{W}^0_h(\Gamma)$ (or merging $\sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \tilde{W}_h(\mathrm{E})$ into $W^0_h(\Gamma)$), the resulting coarse subspace become much larger (compare the BPS preconditioner). If merging $\sum_{\Gamma_{ij}} W_h(\partial\Gamma_{ij})$ (or $\sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \tilde{W}_h(\mathrm{E})$) into $\sum_{\Gamma_{ij}} \tilde{W}_h(\Gamma_{ij})$, it is more difficult to construct efficient local solvers because of the possible large jump of the coefficient (refer to Section 5).*

4 Stable multilevel decomposition for $W_h(\Gamma)$

This section is devoted to establishment of a stable multilevel decomposition for $W_h(\Gamma)$.

4.1 Multilevel decomposition for Γ_{ij}

For a positive integer m_1, we decompose Γ_{ij} into the union of non-overlapping sub-faces $\Gamma^{(1)}_{ij,\ 1}, \cdots, \Gamma^{(1)}_{ij,\ m_1}$, each of which is just the union of some elements. As before, we assume that the decomposition are conforming, and all the sub-faces $\Gamma^{(1)}_{ij,\ r}$ have almost the same "size" d_1. Let J and m_k $(k = 1, \cdots, J)$ be given positive integers. Set $\mathrm{M}_k = m_1 \cdots m_k$ for $k = 1, \cdots, J$. Successively continuing the above procedure, we get the following hierarchical decompositions for Γ_{ij}

• the first level decomposition

$$\bar{\Gamma}_{ij} = \bigcup_{r=1}^{m_1} \bar{\Gamma}^{(1)}_{ij,\ r}.$$

• the second level decomposition

Let each $\Gamma^{(1)}_{ij,\ r}$ be further decomposed into the union of m_2 sub-faces of $\Gamma^{(1)}_{ij,\ r}$

$$\bar{\Gamma}^{(1)}_{ij,\ r} = \bigcup_{l=1}^{m_2} \bar{\Gamma}^{(2)}_{ij,\ m_1(r-1)+l} \quad (r = 1, \cdots, m_1).$$

Thus,

$$\bar{\Gamma}_{ij} = \bigcup_{r=1}^{m_1} \bigcup_{l=1}^{m_2} \bar{\Gamma}^{(2)}_{ij,\ m_2(r-1)+l} = \bigcup_{r=1}^{\mathrm{M}_2} \bar{\Gamma}^{(2)}_{ij,\ r}.$$

• the $k-$th level decomposition for $2 \leq k \leq J$

After generating $\Gamma^{k-1}_{ij,\ r}$ from the $k-1$-level decomposition, we decompose each $\Gamma^{(k-1)}_{ij,\ r}$ into the union of m_k sub-faces of $\Gamma^{(k-1)}_{ij,\ r}$

$$\bar{\Gamma}^{(k-1)}_{ij,\ r} = \bigcup_{l=1}^{m_k} \bar{\Gamma}^{(k)}_{ij,\ m_k(r-1)+l} \quad (r = 1 \cdots \mathrm{M}_{k-1}). \tag{4.1}$$

Finally, we get the multilevel decomposition for Γ_{ij}

$$\bar{\Gamma}_{ij} = \bigcup_{r=1}^{m_1} \bar{\Gamma}^{(1)}_{ij,\ r} = \cdots = \bigcup_{r=1}^{\mathrm{M}_{J-1}} \bigcup_{l=1}^{m_J} \bar{\Gamma}^{(J)}_{ij,\ m_J(r-1)+l} = \bigcup_{r=1}^{\mathrm{M}_J} \bar{\Gamma}^{(J)}_{ij,\ r}.$$

As usual, we assume that $\Gamma^{k}_{ij,\ r}$ has the almost same "size" d_k for each r $(k = 1, \cdots, J)$. Note that M_k denotes the total number of the smallest sub-faces $\Gamma^{(k)}_{ij,r}$ generated from the k-level decomposition for Γ_{ij}.

Remark 4.1 *We do not require that $\Gamma^{(k)}_{ij,\ r}$ is a polygon with finite edges. Thus, the above decomposition can be conveniently generated without any extra restriction to the triangulation on Γ_{ij}. One can also define different (but similar) domain decompositions for Γ_{ij} (and the following space decompositions for $W_h(\Gamma)$).*

4.2 Multilevel decomposition for $W_h(\Gamma)$

For convenience, define $\mathrm{M}_0 = 1$ and $\Gamma^{(0)}_{ij,1} = \Gamma_{ij}$. For $k \geq 1$, let $\mathcal{E}_{\Gamma^{(k-1)}_{ij,\ r}}$ and $\mathcal{F}_{\Gamma^{(k-1)}_{ij,\ r}}$ denote the set of the coarse edges and the set of the sub-faces associated with the decomposition

$$\bar{\Gamma}^{(k-1)}_{ij,\ r} = \bigcup_{l=1}^{m_k} \bar{\Gamma}^{(k)}_{ij,\ m_k(r-1)+l} \quad (r = 1 \cdots \mathrm{M}_{k-1}),$$

respectively. Namely,

$$\mathcal{E}_{\Gamma^{(k-1)}_{ij,\ r}} = \{\partial\Gamma^{(k)}_{ij,\ m_k(r-1)+r} \cap \partial\Gamma^{(k)}_{ij,\ m_k(r-1)+l}:\ 1 \le r,\ l \le m_k,\ r \ne l\},$$

and

$$\mathcal{F}_{\Gamma^{(k-1)}_{ij,\ r}} = \{\Gamma^{(k)}_{ij,\ m_k(r-1)+1}, \Gamma^{(k)}_{ij,\ m_k(r-1)+2}, \cdots, \Gamma^{(k)}_{ij,\ m_k r}\}.$$

Let $\tilde{W}_h(\Gamma^{(0)}_{ij,1}) = \tilde{W}_h(\Gamma_{ij})$ be defined as in subsection 2.2 (for $G = \Gamma_{ij}$). Define

$$\tilde{W}_h(\bar{\Gamma}^k_{ij,\ r}) = \{\tilde{\phi}_h \in \tilde{W}_h(\Gamma_{ij}):\ \tilde{\phi}_h = I^0_{\bar{\Gamma}^k_{ij,\ r}} \phi_h \text{ for some } \phi_h \in \tilde{W}_h(\Gamma_{ij})\}$$

$$(k = 0, \cdots, J;\ r = 1, \cdots, \mathrm{M}_k),$$

and

$$W^0_h(\Gamma^{(k)}_{ij,\ r}) = \{\phi_h \in \tilde{W}_h(\Gamma^{(k)}_{ij,\ r}):\ \phi_h(p) = C_{\mathrm{E}}, \forall p \in \mathcal{N}_h \cap \mathrm{E} \text{ with } \mathrm{E} \in \mathcal{E}_{\Gamma^{(k)}_{ij,\ r}};$$
$$\phi_h(p) = C_{\mathrm{F}}, \forall p \in \mathcal{N}_h \cap \mathrm{F} \text{ with } \mathrm{F} \in \mathcal{F}_{\Gamma^{(k)}_{ij,\ r}}\}$$
$$(k = 0, \cdots, J-1;\ r = 1, \cdots, \mathrm{M}_k).$$

The space $W^0_h(\Gamma^{(k)}_{ij,\ r})$ can be viewed as an extension of $W^0_h(\Gamma)$ defined in subsection 3.2. Note that each function in $W^0_h(\Gamma^{(k)}_{ij,\ r})$ vanishes at all nodes in $\Gamma_{ij} \backslash \bar{\Gamma}^{(k)}_{ij,\ r}$. It is clear that the dimensions of a $W^0_h(\Gamma^{(k)}_{ij,\ r})$ equal to the sum of the number of the sub-faces, the number of internal edges and the number of internal vertices associated with the next decomposition for $\Gamma^{(k)}_{ij,\ r}$. Thus,

$$dim(W^0_h(\Gamma^{(k)}_{ij,\ r})) = O(m_{k+1}) \quad (k = 0, \cdots, J-1;\ r = 1, \cdots, \mathrm{M}_k). \tag{4.2}$$

In the following space decomposition, the spaces $\tilde{W}_h(\bar{\Gamma}^{(J)}_{ij,\ r})$ $(r = 1, \cdots, \mathrm{M}_J)$ will be used as the finest local spaces, and the spaces $W^0_h(\Gamma^{(k)}_{ij,\ r})$ $(r = 1, \cdots, \mathrm{M}_k)$ will be used as the coarse subspaces on the $k+1-$th level decomposition for Γ_{ij} $(k = 0, \cdots, J-1)$. Note that, for a fixed k, the number of such (local) coarse subspaces $W^0_h(\Gamma^{(k)}_{ij,\ r})$ equals to M_k.

For convenience, set

$$W^{00}_h(\Gamma) = \hat{W}^0_h(\Gamma) + \sum_{\Gamma_{ij}} W_h(\partial\Gamma_{ij}) \quad \text{or} \quad W^{00}_h(\Gamma) = W^0_h(\Gamma) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} W_h(\mathrm{E}).$$

Then, we get a multilevel space decomposition for $W_h(\Gamma)$

$$W_h(\Gamma) = W^{00}_h(\Gamma) + \sum_{\Gamma_{ij}} [\sum_{k=0}^{J-1} \sum_{r=1}^{\mathrm{M}_k} W^0_h(\Gamma^{(k)}_{ij,\ r}) + \sum_{r=1}^{\mathrm{M}_J} \tilde{W}_h(\bar{\Gamma}^{(J)}_{ij,\ r})].$$

4.3 Main result

Before giving the main result, we prove a lemma.

Let k be a positive integer satisfying $1 \le k \le J$. For a node $p \in \mathcal{N}_h \cap \bar{\Gamma}^{(k-1)}_{ij,\ r}$ with $r = 1, \cdots, \mathrm{M}_{k-1}$, let $n^{(k-1)}_{ij,\ r}(p)$ denote the number of the sub-faces $\Gamma^{(k)}_{ij,\ m_k(r-1)+l}$ containing p. For example, when p is an internal node of an edge in $\mathcal{E}_{\Gamma^{(k-1)}_{ij,\ r}}$, we have $n^{(k-1)}_{ij,\ r}(p) = 2$. For $l = 1, \cdots, m_k$, define the extension operator

$$\theta^{(k)}_{ij,\ m_k(r-1)+l} : W_h(\Gamma) \to \tilde{W}_h(\bar{\Gamma}^{(k)}_{ij,\ m_k(r-1)+l}) \subset \tilde{W}_h(\bar{\Gamma}^{(k-1)}_{ij,\ r})$$

by

$$(\theta^{(k)}_{ij,\ m_k(r-1)+l}\phi_h)(p)=\begin{cases}\frac{1}{n^{(k-1)}_{ij,\ r}(p)}\phi_h(p), & if\ p\in\mathcal{N}_h\cap(\bar{\Gamma}^{(k)}_{ij,\ m_k(r-1)+l}\backslash\partial\Gamma^{(k-1)}_{ij,\ r}),\\ 0, & \text{if } p\in\mathcal{N}_h\cap(\partial\Gamma^{(k-1)}_{ij,\ r}\cup\Gamma\backslash\bar{\Gamma}^{(k)}_{ij,\ m_k(r-1)+l}),\end{cases}\quad(\phi_h\in W_h(\Gamma)).$$

Such extension operators satisfy

$$\sum_{l=1}^{m_k}\theta^{(k)}_{ij,\ m_k(r-1)+l}=\mathrm{I}\quad\text{on}\quad\tilde{W}_h(\bar{\Gamma}^{(k-1)}_{ij,\ r}).\tag{4.3}$$

For $\phi_h\in W_h(\Gamma)$, let $\phi_{ij}\in\tilde{W}_h(\Gamma_{ij})$ be defined in the proofs of Theorem 3.1 or of Theorem 3.2. For convenience, set $\phi^{(0)}_{ij,1}=\phi_{ij}$. Define

$$\phi^{(k)}_{ij,\ m_k(r-1)+l}=\theta^{(k)}_{ij,\ m_k(r-1)+l}[\phi_h-\gamma_{\Gamma^{(k)}_{ij,\ m_k(r-1)+l}}(\phi_h)]\in\tilde{W}_h(\bar{\Gamma}^{(k)}_{ij,\ m_k(r-1)+l})\tag{4.4}$$

$$(1\leq k\leq J;\ r=1,\cdots,\mathrm{M}_{k-1};\ l=1,\cdots,m_k),$$

and

$$\phi^{(k-1)}_{ij,\ r,\ 0}=\phi^{(k-1)}_{ij,\ r}-\sum_{l=1}^{m_k}\phi^{(k)}_{ij,\ m_k(r-1)+l}\quad(1\leq k\leq J;\ r=1,\cdots,\mathrm{M}_{k-1}).\tag{4.5}$$

Lemma 4.1 *Let $\phi^{(k)}_{ij,\ m_k(r-1)+l}$ and $\phi^{(k-1)}_{ij,\ r,\ 0}$ be defined above. Then,*

$$\begin{aligned}\sum_{r=1}^{\mathrm{M}_{k-1}}\|\phi^{(k)}_{ij,\ m_k(r-1)+l}\|^2_{*,\Gamma}&=\sum_{r=1}^{\mathrm{M}_{k-1}}\sum_{l=1}^{m_k}\|\phi^{(k)}_{ij,\ m_k(r-1)+l}\|^2_{*,\Gamma}\\&\lesssim[1+\log(d_k/h)]^2\|\phi_h\|^2_{*,\Gamma}\quad(1\leq k\leq J).\end{aligned}\tag{4.6}$$

Moreover, we have

$$\phi^{(k-1)}_{ij,\ r,\ 0}\in W^0_h(\Gamma^{(k-1)}_{ij,\ r})\quad(1\leq k\leq J;\ r=1,\cdots,\mathrm{M}_{k-1}).\tag{4.7}$$

Proof. Let $r=1,\cdots,\mathrm{M}_{k-1};\ l=1,\cdots,m_k$. Using (2.10) with $\mathrm{F}=\Gamma^{(k)}_{ij,\ m_k(r-1)+l}$, one can verify that (refer to the last section)

$$\begin{aligned}\|\phi^{(k)}_{ij,\ m_k(r-1)+l}\|^2_{*,\Gamma}&\lesssim[1+\log(d_k/h)]^2(\omega_i+\omega_j)\|\phi_h-\gamma_{\Gamma^{(k)}_{ij,\ m_k(r-1)+l}}(\phi_h)\|^2_{\frac{1}{2},\Gamma^{(k)}_{ij,\ m_k(r-1)+l}}\\&\lesssim[1+\log(d_k/h)]^2(\omega_i+\omega_j)|\phi_h|^2_{\frac{1}{2},\Gamma^{(k)}_{ij,\ m_k(r-1)+l}}.\end{aligned}$$

This, together with (4.1), leads to

$$\sum_{l=1}^{m_k}\|\phi^{(k)}_{ij,\ m_k(r-1)+l}\|^2_{*,\Gamma}\lesssim[1+\log(d_k/h)]^2(\omega_i+\omega_j)|\phi_h|^2_{\frac{1}{2},\Gamma^{(k-1)}_{ij,\ r}}.$$

We further get (4.6).

On the other hand, we have by (4.5) and (4.3)

$$\phi^{(k-1)}_{ij,\ r,\ 0}=\sum_{l=1}^{m_k}\theta^{(k)}_{ij,\ m_k(r-1)+l}[\gamma_{\Gamma^{(k)}_{ij,\ m_k(r-1)+l}}(\phi_h)-\gamma_{\Gamma^{(k-1)}_{ij,\ r}}(\phi_h)],$$

which implies (4.7).

□

For a $\phi_h\in W_h(\Gamma)$, let $\hat{\phi}_0,\phi^{\partial}_{ij},\phi_0$ and ϕ_{E} be defined by the last subsection. For simplicity of exposition, we define

$$\phi_{00}=\hat{\phi}_0+\sum_{\Gamma_{ij}}\phi^{\partial}_{ij}\quad\text{or}\quad\phi_{00}=\phi_0+\sum_{\mathrm{E}\in\mathcal{E}_\Gamma}\phi_{\mathrm{E}}.$$

Theorem 4.1 *For any $\phi_h \in W_h(\Gamma)$, let $\phi_{00} \in W_h^{00}(\Gamma)$ be defined above. Then, there exist functions*

$\phi_{ij,\ r,\ 0}^{(k)} \in W_h^0(\Gamma_{ij,\ r}^{(k)})$ $(0 \leq k \leq J-1;\ r = 1, \cdots, \mathrm{M}_k)$ *and* $\phi_{ij,\ r}^{(J)} \in \tilde{W}_h(\bar{\Gamma}_{ij,\ r}^{(J)})$ $(r = 1, \cdots, \mathrm{M}_J)$,

such that

$$\phi_h = \phi_{00} + \sum_{\Gamma_{ij}} [\sum_{k=0}^{J-1} \sum_{r=1}^{\mathrm{M}_k} \phi_{ij,\ r\ ,0}^{(k)} + \sum_{r=1}^{\mathrm{M}_J} \phi_{ij,\ r}^{(J)}]. \tag{4.8}$$

Moreover, we have

$$\sum_{\Gamma_{ij}} [\sum_{k=0}^{J-1} \sum_{r=1}^{\mathrm{M}_k} \|\phi_{ij,\ r\ ,0}^{(k)}\|_{*,\Gamma}^2 + \sum_{r=1}^{\mathrm{M}_J} \|\phi_{ij,\ r}^{(J)}\|_{*,\Gamma}^2] \lesssim J[1 + \log(H/h)]^2 \|\phi_h\|_{*,\Gamma}^2 \quad (J \geq 1). \tag{4.9}$$

Proof. The decompositions (3.4) and (3.14) can be written as

$$\phi_h = \phi_{00} + \sum_{\Gamma_{ij}} \phi_{ij}. \tag{4.10}$$

For $k \geq 1$, let $\phi_{ij,\ r,\ 0}^{(k)} \in W_h^0(\Gamma_{ij,\ r}^{(k)})$ be defined by (4.5). Then, we have by (4.5) for $k = 1$ and $k = 2$

$$\phi_{ij} = \phi_{ij,\ 1,\ 0}^{(0)} + \sum_{r=1}^{m_1} \phi_{ij,\ r}^{(1)} = \phi_{ij,\ 1,\ 0}^{(0)} + \sum_{r=1}^{m_1} \phi_{ij,\ r,\ 0}^{(1)} + \sum_{r=1}^{m_1} \sum_{l=1}^{m_2} \phi_{ij,\ m_2(r-1)+l}^{(2)}. \tag{4.11}$$

It is easy to see that

$$\sum_{r=1}^{m_1} \sum_{l=1}^{m_2} \phi_{ij,\ m_2(r-1)+l}^{(2)} = \sum_{r=1}^{\mathrm{M}_2} \phi_{ij,\ r}^{(2)}.$$

Substituting this into (4.11), and using (4.5) repeatedly, yields

$$\begin{aligned}
\phi_{ij} &= \phi_{ij,\ 1,\ 0}^{(0)} + \sum_{r=1}^{m_1} \phi_{ij,\ r,\ 0}^{(1)} + \sum_{r=1}^{\mathrm{M}_2} \phi_{ij,\ r,\ 0}^{(2)} + \sum_{r=1}^{\mathrm{M}_3} \phi_{ij,\ r}^{(3)} = \cdots \\
&= \phi_{ij,\ 1,\ 0}^{(0)} + \sum_{r=1}^{m_1} \phi_{ij,\ r,\ 0}^{(1)} + \cdots + \sum_{r=1}^{\mathrm{M}_{J-1}} \phi_{ij,\ r,\ 0}^{(J-1)} + \sum_{r=1}^{\mathrm{M}_J} \phi_{ij,\ r}^{(J)} \\
&= \sum_{k=0}^{J-1} \sum_{r=1}^{\mathrm{M}_k} \phi_{ij,\ r\ ,0}^{(k)} + \sum_{r=1}^{\mathrm{M}_J} \phi_{ij,\ r}^{(J)}.
\end{aligned}$$

Plugging this in (4.10), gives (4.8).

On the other hand, it follows by (4.5) that

$$\begin{aligned}
\|\phi_{ij,\ r,\ 0}^{(k-1)}\|_{*,\ \Gamma}^2 &\leq 2(\|\phi_{ij,\ r}^{(k-1)}\|_{*,\Gamma}^2 + \|\sum_{l=1}^{m_k} \phi_{ij,\ m_k(r-1)+l}^{(k)}\|_{*,\Gamma}^2) \\
&\lesssim \|\phi_{ij,\ r}^{(k-1)}\|_{*,\Gamma}^2 + \sum_{l=1}^{m_k} \|\phi_{ij,\ m_k(r-1)+l}^{(k)}\|_{*,\Gamma}^2.
\end{aligned} \tag{4.12}$$

Here, we have used the fact that each $\phi_{ij,\ m_k(r-1)+l}^{(k)}$ has local support. For convenience, we set $d_0 = H$. Combining (4.12) with (4.6), leads to

$$\sum_{r=1}^{\mathrm{M}_{k-1}} \|\phi_{ij,\ r,\ 0}^{(k-1)}\|_{*,\Gamma}^2 \leq \sum_{r=1}^{\mathrm{M}_{k-1}} \|\phi_{ij,\ r}^{(k-1)}\|_{*,\Gamma}^2 + \sum_{r=1}^{\mathrm{M}_{k-1}} \sum_{l=1}^{m_k} \|\phi_{ij,\ m_k(r-1)+l}^{(k)}\|_{*,\Gamma}^2)$$

$$\lesssim \; [1+\log(d_{k-1}/h)]^2 \|\phi_h\|_{*,\Gamma}^2 \;\; (k=1,\cdots,J).$$

Thus,

$$\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k} \|\phi_{ij,\ r\ ,0}^{(k)}\|_{*,\Gamma}^2 \lesssim (\sum_{k=0}^{J-1}[1+\log(d_k/h)]^2)\|\phi_h\|_{*,\Gamma}^2 \lesssim J[1+\log(H/h)]^2)\|\phi_h\|_{*,\Gamma}^2. \quad (4.13)$$

Now, the inequality (4.9) follows by (4.13) and (4.6) with $k=J$.

□

5 Multilevel preconditioner for S_h

In this section, we construct a kind of multilevel preconditioner for S_h based on the multilevel decomposition introduced in the last section.

5.1 Inexact coarse solver

In the following, we consider only the second basic decomposition defined in Subsection 3.2. One needs to construct a symmetric and positive definite operator $M_0 : W_h^0(\Gamma) \rightarrow W_h^0(\Gamma)$, such that M_0 is spectrally equivalent to the restriction of S_h on $W_h^0(\Gamma)$. Namely,

$$\langle M_0\varphi_h, \varphi_h\rangle_\Gamma \eqsim \|\varphi_h\|_{*,\Gamma}^2, \quad \forall \varphi_h \in W_h^0(\Gamma). \quad (5.1)$$

For a $\partial\Omega_k$, let $\mathcal{N}_{h,k}$ denote the set of nodes on $\partial\Omega_k$. Define the discrete $H^{\frac{1}{2}}$-inner product $\langle\cdot,\cdot\rangle_{h,\partial\Omega_k}$ as

$$\langle\varphi_h,\psi_h\rangle_{h,\partial\Omega_k} = h^4 \sum_{\substack{p,q\in\mathcal{N}_{h,k}\\ p\neq q}} \frac{(\varphi_h(p)-\varphi_h(q))(\psi_h(p)-\psi_h(q))}{|p-q|^3}, \quad \varphi_h,\ \psi_h \in W_h(\partial\Omega_k).$$

Then, the preconditioner M_0 can be defined by

$$\langle M_0\varphi_h,\psi_h\rangle_\Gamma = \sum_{k=1}^{\mathrm{N}} \omega_k \langle\varphi_h,\psi_h\rangle_{h,\partial\Omega_k} \quad \varphi_h \in W_h^0(\Gamma),\ \forall\psi_h \in W_h^0(\Gamma).$$

In the following, we explain how to compute the stiffness matrix of M_0.

Let $\{\varphi_i^0\}$ denote the basis of W_0. Note that either $\varphi_i^0(p)=1$ or $\varphi_i^0(p)=0$ for each node p. Set

$$\mathcal{V}_i^k = \{p\in\mathcal{N}_h\cap\partial\Omega_k : \ \varphi_i^0(p)=1\}.$$

It can be verified that

$$\langle\varphi_r^0,\varphi_l^0\rangle_{h,\partial\Omega_k} = \begin{cases} h^4 \sum\limits_{\substack{p\in\mathcal{V}_r^k, q\in\mathcal{V}_l^k\\ p\neq q}} \frac{-1}{|p-q|^3}, & \text{if } r\neq l,\\ h^4 \sum\limits_{p\in\mathcal{V}_r^k}\sum\limits_{q\notin\mathcal{V}_r^k} \frac{1}{|p-q|^3}, & \text{if } r=l.\end{cases}$$

For convenience, set $a_{rl}^k = \langle\varphi_r^0,\varphi_l^0\rangle_{h,\partial\Omega_k}$, and define

$$\mathcal{Q}^{rl} = \{k: \text{ both } \mathcal{V}_r^k\neq\emptyset \text{ and } \mathcal{V}_l^k\neq\emptyset\}.$$

Then,

$$\langle M_0\varphi_r^0,\varphi_l^0\rangle_\Gamma = \sum_{k\in\mathcal{Q}^{rl}} \omega_k a_{rl}^k.$$

It is easy to see that $\langle M_0\varphi_r^0,\varphi_l^0\rangle_\Gamma\neq 0$ if and only if $\mathcal{Q}^{rl}$ is not an empty set. Thus, the stiffness matrix of M_0 is a sparse and band matrix. Moreover, it is not expensive to get such stiffness matrix.

5.2 Preconditioner for Steklov-Poincar$\acute{e}$ operators

Consider a local interface Γ_{ij}. Let $S_{ij}: \tilde{W}_h(\Gamma_{ij}) \rightarrow \tilde{W}_h(\Gamma_{ij})$ denote (discrete) Steklov-Poincar$\acute{e}$ operators defined as usual. Namely, $S_{ij} = S_h|_{\tilde{W}_h(\Gamma_{ij})}$. One needs to construct a suitable symmetric and positive definite operator $M_{ij}: \tilde{W}_h(\Gamma_{ij}) \rightarrow \tilde{W}_h(\Gamma_{ij})$, such that not only the action of M_{ij}^{-1} is cheaper to implement than that of S_{ij}^{-1}, but also

$$\langle M_{ij}\phi_h, \phi_h\rangle_{\Gamma_{ij}} \eqsim (\omega_i + \omega_j)|\phi_h|^2_{H^{\frac{1}{2}}_{00}(\Gamma_{ij})}, \quad \forall \phi_h \in \tilde{W}_h(\Gamma_{ij}). \tag{5.2}$$

An interesting method has been developed for preconditioning Poincar$\acute{e}$-Steklov operator (see [17], [18] and [21]). The main idea is to define the inverse of the preconditioner directly, instead of the preconditioner itself. In this method, only matrix-vector multiplication is needed, but the inverse of the preconditioner is not involved. Thus, the cost of computation is decreased in a simple manner. In this subsection, we apply this idea to constructing M_{ij}.

It can be verified that (5.2) is equivalent to

$$\langle M_{ij}^{-1}\varphi_h, \varphi_h\rangle_{\Gamma_{ij}} \eqsim (\omega_i + \omega_j)^{-1}\|\varphi_h\|^2_{-\frac{1}{2},\Gamma_{ij}}, \quad \forall \varphi_h \in \tilde{W}_h(\Gamma_{ij}). \tag{5.3}$$

Thus, the preconditioner M_{ij} needs only to satisfy (5.3). In the following, we use a boundary integral operator to construct M_{ij}.

Let G be the fundamental solution of the Laplacian, i.e.,

$$G(p,q) = \frac{1}{4\pi}\frac{1}{|p-q|}, \quad p, q \in \mathcal{R}^3.$$

Define the operator

$$V_{ij}\varphi(q) = \int_{\Gamma_{ij}} G(p,q)\varphi(p)dp, \quad q \in \Gamma_{ij}.$$

It is known that V_{ij} is an isomorphism from $H^{-\frac{1}{2}}(\Gamma_{ij})$ into $H^{\frac{1}{2}}_{00}(\Gamma_{ij})$, and

$$\langle V_{ij}\varphi, \varphi\rangle_{\Gamma_{ij}} \eqsim \|\varphi\|^2_{-\frac{1}{2},\Gamma_{ij}}, \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma_{ij}).$$

A natural way is to choose M_{ij}^{-1} as the restriction of $(\omega_i + \omega_j)^{-1}V_{ij}$ on $\tilde{W}_h(\Gamma_{ij})$. Namely, we define M_{ij}^{-1} by

$$\begin{aligned}\langle M_{ij}^{-1}\varphi_h, \psi_h\rangle_{\Gamma} &= \frac{1}{4\pi(\omega_i + \omega_j)}\int_{\Gamma_{ij}}\int_{\Gamma_{ij}} \frac{\varphi_h(p)\psi_h(q)}{|p-q|}ds(p)ds(q), \\ &\qquad \varphi_h \in \tilde{W}_h(\Gamma_{ij}),\ \forall \psi_h \in \tilde{W}_h(\Gamma_{ij}).\end{aligned} \tag{5.4}$$

5.3 Preconditioner for local operators

We first consider preconditioner on $W_h(\mathrm{E})$.

Let $\mathcal{Q}_{\mathrm{E}}$ be defined by subsection 2.2. It can be verified that

$$\|\varphi_h\|^2_{*,\Gamma} \eqsim (\sum_{k \in \mathcal{Q}_{\mathrm{E}}} \omega_k)\|\varphi_h\|^2_{0,\mathrm{E}}, \quad \forall \varphi_h \in \tilde{W}_h(\mathrm{E}).$$

This means that a preconditioner M_{E} for $S_h|_{W_h(\mathrm{E})}$ should satisfies

$$\langle M_{\mathrm{E}}\varphi_h, \varphi_h\rangle_{\mathrm{E}} \eqsim (\sum_{k \in \mathcal{Q}_{\mathrm{E}}} \omega_k)\|\varphi_h\|^2_{0,\mathrm{E}}, \quad \forall \varphi_h \in \tilde{W}_h(\mathrm{E}), \tag{5.5}$$

which is equivalent to

$$\langle M_{\rm E}^{-1}\varphi_h,\varphi_h\rangle_{\rm E} \eqsim (\sum_{k\in\mathcal{Q}_{\rm E}}\omega_k)^{-1}\|\varphi_h\|_{0,{\rm E}}^2,\quad \forall\varphi_h\in\tilde{W}_h({\rm E}).$$

Thus, we define the preconditioner $M_{\rm E}$ by

$$\langle M_{\rm E}^{-1}\varphi_h,\psi_h\rangle_{\rm E} \eqsim (\sum_{k\in\mathcal{Q}_{\rm E}}\omega_k)^{-1}\langle\varphi_h,\psi_h\rangle_{\rm E},\quad \varphi_h\in\tilde{W}_h({\rm E}),\ \forall\psi_h\in\tilde{W}_h({\rm E}). \tag{5.6}$$

Consider a space $W_h^0(\Gamma_{ij,\ r}^{(k)})$ ($k=0,\cdots,J-1;\ r=1,\cdots,{\rm M}_k$). One needs to construct a preconditioner for the restriction of S_h on $W_h^0(\Gamma_{ij,\ r}^{(k)})$. Let $M_{ij,\ r,\ 0}^{(k)}$ be such a preconditioner. Then, $M_{ij,\ r,\ 0}^{(k)}$ should satisfy

$$\langle M_{ij,\ r,\ 0}^{(k)}\varphi_h,\varphi_h\rangle_{\Gamma_{ij,\ r}^{(k)}} \eqsim (\omega_i+\omega_j)|\varphi_h|^2_{H_{00}^{\frac{1}{2}}(\Gamma_{ij,\ r}^{(k)})},\quad \forall\varphi_h\in W_h^0(\Gamma_{ij,\ r}^{(k)}). \tag{5.7}$$

Namely,

$$\langle (M_{ij,\ r,\ 0}^{(k)})^{-1}\varphi_h,\varphi_h\rangle_{\Gamma_{ij,\ r}^{(k)}} \eqsim (\omega_i+\omega_j)^{-1}|\varphi_h|^2_{-\frac{1}{2},\ \Gamma_{ij,\ r}^{(k)}},\quad \forall\varphi_h\in W_h^0(\Gamma_{ij,\ r}^{(k)}).$$

Thus, one needs only to define $(M_{ij,\ r,\ 0}^{(k)})^{-1}$ as the restriction of $(\omega_i+\omega_j)^{-1}V_{ij}$ on $W_h^0(\Gamma_{ij,\ r}^{(k)})$. Namely,

$$\langle (M_{ij,\ r,\ 0}^{(k)})^{-1}\varphi_h,\psi_h\rangle_{\Gamma_{ij,\ r}^{(k)}} = \frac{1}{4\pi(\omega_i+\omega_j)}\int_{\Gamma_{ij,\ r}^{(k)}}\int_{\Gamma_{ij,\ r}^{(k)}}\frac{\varphi_h(p)\psi_h(q)}{|p-q|}ds(p)ds(q),$$
$$\varphi_h\in W_h^0(\Gamma_{ij,\ r}^{(k)}),\ \forall\psi_h\in W_h^0(\Gamma_{ij,\ r}^{(k)}). \tag{5.8}$$

Similarly, we can define a preconditioner $M_{ij,\ r}^{(J)}$ for the restriction of S_h on $\tilde{W}_h(\Gamma_{ij,\ r}^{J})$ by

$$\langle (M_{ij,\ r}^{(J)})^{-1}\varphi_h,\psi_h\rangle_{\Gamma_{ij,\ r}^{J}} = \frac{1}{4\pi(\omega_i+\omega_j)}\int_{\Gamma_{ij,\ r}^{(J)}}\int_{\Gamma_{ij,\ r}^{(J)}}\frac{\varphi_h(p)\psi_h(q)}{|p-q|}ds(p)ds(q),$$
$$\varphi_h\in\tilde{W}_h(\Gamma_{ij,\ r}^{(J)}),\ \forall\psi_h\in\tilde{W}_h(\Gamma_{ij,\ r}^{(J)}). \tag{5.9}$$

Remark 5.1 *For the new multilevel method, it is not essential to use the integral operator V_{ij} for the definition of M_{ij}^{-1}, $(M_{ij,\ r,\ 0}^{(k)})^{-1}$ ($k=0,\cdots,J-1$) and $(M_{ij,\ r}^{(J)})^{-1}$. One can also define the stiffness matrices of M_{ij}, $M_{ij,\ r,\ 0}^{(k)}$ ($k=0,\cdots,J-1$) and $M_{ij,\ r}^{(J)}$ directly by the discrete norm associated with $H_{00}^{\frac{1}{2}}(\Gamma_{ij})$ (refer to Subsection 5.1).*

5.4 Preconditioners

Now, we can describe the preconditioner for S_h.

As usual, we define some L^2 projectors:

$$Q_0: W_h^0(\Gamma)\to W_h^0(\Gamma),\quad Q_{ij}:\tilde{W}_h(\Gamma_{ij})\to\tilde{W}_h(\Gamma_{ij}),\quad Q_{\rm E}: W_h({\rm E})\to W_h({\rm E}),$$

and

$$Q_{ij,\ r,\ 0}^{(k)}: W_h^0(\Gamma_{ij,\ r}^{(k)})\to W_h^0(\Gamma_{ij,\ r}^{(k)}),\quad Q_{ij,\ r}^{(J)}:\tilde{W}_h(\Gamma_{ij,\ r}^{(J)})\to\tilde{W}_h(\Gamma_{ij,\ r}^{(J)}).$$

Then, the preconditioner can be defined as follows

- One-level preconditioner

$$M^{-1} = M_0^{-1}Q_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} M_{\mathrm{E}}^{-1}Q_{\mathrm{E}} + \sum_{\Gamma_{ij}} M_{ij}^{-1}Q_{ij}.$$

• Multilevel preconditioner

$$M_J^{-1} = M_0^{-1}Q_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} M_{\mathrm{E}}^{-1}Q_{\mathrm{E}} + \sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}(M_{ij,\ r,\ 0}^{(k)})^{-1}Q_{ij,\ r,\ 0}^{(k)} + \sum_{r=1}^{\mathrm{M}_J}(M_{ij,\ r}^{(J)})^{-1}Q_{ij,\ r}^{(J)}].$$

Theorem 5.1 *For the preconditioners defined above, we have*

$$cond(M^{-1}S_h) \le C[1+\log(H/h)]^2, \tag{5.10}$$

and

$$cond(M_J^{-1}S_h) \le CJ^2[1+\log(H/h)]^2. \tag{5.11}$$

Hereafter, C is a constant independent of h, H, d_k and the jumps of the coefficient $\omega(p)$ across the interface.

Proof. The estimate (5.10) can be verified in the standard way, together with Theorem 3.1 and Theorem 3.2. One needs only to consider (5.11). By the convergence theory of interface preconditioner (see [20] and [33]), together with Theorem 4.1, it suffices to prove that

$$\|\varphi_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma}\varphi_{\mathrm{E}} + \sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}\varphi_{ij,\ r,\ 0}^{(k)} + \sum_{r=1}^{\mathrm{M}_J}\tilde{\varphi}_{ij,\ r}^{(J)}]\|_{*,\ \Gamma}^2 \lesssim J\{\langle M_0\varphi_0,\varphi_0\rangle + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma}\langle M_{\mathrm{E}}\varphi_{\mathrm{E}},\ \varphi_{\mathrm{E}}\rangle_{\mathrm{E}}$$

$$+\sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}\langle M_{ij,\ r,\ 0}^{(k)}\varphi_{ij,\ r,\ 0}^{(k)},\ \varphi_{ij,\ r,\ 0}^{(k)}\rangle_{\Gamma_{ij,\ r}^{(k)}} + \sum_{r=1}^{\mathrm{M}_J}\langle M_{ij,\ r}^{(J)}\tilde{\varphi}_{ij,\ r}^{(J)},\ \tilde{\varphi}_{ij,\ r}^{(J)}\rangle_{\Gamma_{ij,\ r}^{(J)}}]\},$$

$$\forall\varphi_0\in W_h^0(\Gamma);\ \varphi_{\mathrm{E}}\in W_h(\mathrm{E});\ \varphi_{ij,\ r,\ 0}^{(k)}\in W_h^0(\Gamma_{ij,\ r,\ 0}^{(k)});\ \tilde{\varphi}_{ij,\ r}^{(J)}\in\tilde{W}_h(\Gamma_{ij,\ r}^{(J)}).$$

Note that there are only two subdomains containing Γ_{ij}, the above inequality can be derived by (5.1), (5.5) and

$$\begin{aligned}\|\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}\varphi_{ij,\ r,\ 0}^{(k)} + \sum_{r=1}^{\mathrm{M}_J}\tilde{\varphi}_{ij,\ r}^{(J)}\|_{*,\ \Gamma}^2 \quad&\lesssim\quad J[\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}\langle M_{ij,\ r,\ 0}^{(k)}\varphi_{ij,\ r,\ 0}^{(k)},\ \varphi_{ij,\ r,\ 0}^{(k)}\rangle_{\Gamma_{ij,\ r}^{(k)}}\\ &+\quad \sum_{r=1}^{\mathrm{M}_J}\langle M_{ij,\ r}^{(J)}\tilde{\varphi}_{ij,\ r}^{(J)},\ \tilde{\varphi}_{ij,\ r}^{(J)}\rangle_{\Gamma_{ij,\ r}^{(J)}}].\end{aligned} \tag{5.12}$$

It is easy to see that

$$\|\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}\varphi_{ij,\ r,\ 0}^{(k)} + \sum_{r=1}^{\mathrm{M}_J}\tilde{\varphi}_{ij,\ r}^{(J)}\|_{*,\ \Gamma}^2 \lesssim J\sum_{k=0}^{J-1}\|\sum_{r=1}^{\mathrm{M}_k}\varphi_{ij,\ r,\ 0}^{(k)}\|_{*,\ \Gamma}^2 + \|\sum_{r=1}^{\mathrm{M}_J}\tilde{\varphi}_{ij,\ r}^{(J)}\|_{*,\ \Gamma}^2. \tag{5.13}$$

For a fixed k, the sub-faces $\Gamma_{ij,\ r}^{(k)}$ are non-overlapping each other (for different r). Thus, we have by (5.7)

$$\|\sum_{r=1}^{\mathrm{M}_k}\varphi_{ij,\ r,\ 0}^{(k)}\|_{*,\ \Gamma}^2 \quad=\quad (\omega_i+\omega_j)|\sum_{r=1}^{\mathrm{M}_k}\varphi_{ij,\ r,\ 0}^{(k)}|_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})}^2$$

$$
\begin{aligned}
&\lesssim \sum_{r=1}^{\mathrm{M}_k} |\varphi^{(k)}_{ij,\ r,\ 0}|^2_{H^{\frac{1}{2}}_{00}(\Gamma^{(k)}_{ij,\ r})} \\
&\lesssim \langle M^{(k)}_{ij,\ r,\ 0}\varphi^{(k)}_{ij,\ r,\ 0},\ \varphi^{(k)}_{ij,\ r,\ 0}\rangle_{\Gamma^{(k)}_{ij,\ r}}. \qquad (5.14)
\end{aligned}
$$

Similarly, we have by (5.9)

$$
\|\sum_{r=1}^{\mathrm{M}_J}\tilde{\varphi}^{(J)}_{ij,\ r}\|^2_{*,\ \Gamma} \lesssim \langle M^{(J)}_{ij,\ r}\tilde{\varphi}^{(J)}_{ij,\ r},\ \tilde{\varphi}^{(J)}_{ij,\ r}\rangle_{\Gamma^{(J)}_{ij,\ r}}.
$$

Plugging (5.14) and the above inequality in (5.13), gives (5.12).

□

Remark 5.2 *We conjecture that the factor J in (4.9) can be disappeared. In this case, the factor J^2 in (5.11) should be replaced by J. Unfortunately, we fails to prove this conjecture.*

5.5 Implementation and computational complexity

The action of M_J^{-1} can be implemented in the standard way, provided that the stiffness matrices of V_{ij} are gotten on all local subspaces. The calculation of such stiffness matrices has been studied in the boundary element methods by many researchers, see, for example, [11], [13] and [30]. Since V_{ij} is a weakly singular integral operator, we can simplify the calculation of such integrations in a suitable manner, especially for the basis functions of $W_h^0(\Gamma^{(k)}_{ij,\ r})$.

For ease of notation, set

$$
\hat{M}_{ij}^{-1} = M_{ij}^{-1}Q_{ij},
$$

and

$$
\hat{M}_{ij}^{-1}(J) = \sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}(M^{(k)}_{ij,\ r,\ 0})^{-1}Q^{(k)}_{ij,\ r,\ 0} + \sum_{r=1}^{\mathrm{M}_J}(M^{(J)}_{ij,\ r})^{-1}Q^{(J)}_{ij,\ r}.
$$

We compare the computational complexity for implementing the action of $\hat{M}_{ij}^{-1}$ and of $\hat{M}_{ij}^{-1}(J)$.

Let n_{ij} denote the number of the nodes on Γ_{ij}. It is clear that $n_{ij} = O((\frac{H}{h})^2)$. Then, the complexity for computing M_{ij}^{-1} equals to $O(n_{ij}^2)$, and so the complexity for computing $\hat{M}_{ij}^{-1}$ equals to $O(n_{ij}^2)$.

Let $N_{ij}(J)$ denote the computational complexity for implementing the action of $\hat{M}_{ij}^{-1}(J)$.
Proposition 5.1. Let $m \geq 2$ be a given positive integer. Set

$$
J \eqsim \log_2(\log_m n_{ij} + \frac{1}{2}), \qquad (5.15)
$$

and choose m_k by

$$
m_J = m, \quad m_k = m^{2^{J-k}} \quad (k = 1, \cdots, J-1). \qquad (5.16)
$$

Then,

$$
N_{ij}(J) \lesssim \log_2(\log_m n_{ij} + \frac{1}{2}) \cdot n_{ij}, \qquad (5.17)
$$

which is almost optimal.
Proof. It is easy to see that the dimensions of $\tilde{W}_h(\Gamma^{(J)}_{ij,\ r})$ equal to n_{ij}/M_J. This, together with (4.2), leads to

$$
N_{ij}(J) \quad \lesssim \quad m_1^2 + m_1 \cdot m_2^2 + \cdots + \mathrm{M}_{J-1} \cdot m_J^2 + \mathrm{M}_J \cdot (\frac{n_{ij}}{\mathrm{M}_J})^2
$$

$$= \quad m_1^2 + m_1 \cdot m_2^2 + \cdots + \mathrm{M}_{J-1} \cdot m_J^2 + \frac{n_{ij}^2}{\mathrm{M}_J}. \tag{5.18}$$

Using (5.16) and (5.15), yields

$$m_1^2 = m_1 \cdot m_2^2 = \cdots = \mathrm{M}_{J-1} \cdot m_J^2 = m^{2^J} = m \cdot n_{ij}.$$

Moreover, one can verify, by (5.15), that

$$\frac{n_{ij}^2}{\mathrm{M}_J} = \frac{n_{ij}^2}{m^{2^J-1}} \eqsim m \cdot n_{ij}.$$

Substituting the above two inequalities into (5.18), we get

$$N_{ij}(J) \lesssim J \cdot m \cdot n_{ij}.$$

Note that m is a constant, we further deduce (5.17) by this, together with (5.15).
□

The idea for choosing J and m_k is to guarantee that

$$m_1^2 = m_1 \cdot m_2^2 = \cdots = \mathrm{M}_{J-1} \cdot m_J^2 \eqsim \frac{n_{ij}^2}{\mathrm{M}_J}, \tag{5.19}$$

which makes the right side of (5.18) reaches its minimal value. It follows by (5.19) that

$$m_k \eqsim n_{ij}^{\frac{2^{J-k+1}}{2^{J+1}-1}} \quad (k = 1, \cdots, J). \tag{5.20}$$

This, together with (5.18), leads to

$$N_{ij}(J) \lesssim J \cdot m_1^2 \eqsim J \cdot n_{ij}^{\frac{2^{J+1}}{2^{J+1}-1}}. \tag{5.21}$$

Examples:
(a) when $J = 1$, we have by (5.20) and (5.21)

$$m_1 = [n_{ij}^{\frac{2}{3}}], \quad \text{and} \quad N_{ij}(1) = O(n_{ij}^{\frac{4}{3}});$$

(b) when $J = 2$, we have

$$m_1 = [n_{ij}^{\frac{4}{7}}], \quad m_2 = [n_{ij}^{\frac{2}{7}}] \quad \text{and} \quad N_{ij}(2) = O(n_{ij}^{\frac{8}{7}}).$$

Remark 5.3 *From Proposition 5.1, we know that the number J in (5.11) is much less than* $\log(H/h)$*. For a given positive $m \geq 2$, one can set $J \eqsim \log_m(H/h)$, and choose $m_1 = m_2 = \cdots = m_J = m$. For this case, we have $N_{ij}(J) = O(n_{ij})$. Since the action of $\hat{M}_{ij}^{-1}(J)$ can be implemented in parallel, it is also satisfactory to choose a small positive integer J only (for example, set $J = 2$). For a small J, the complexity $N_{ij}(J)$ is still much less than $O(n_{ij}^2)$.*

Remark 5.4 *There are two other classes of multilevel methods in the literature. The one is the multilevel method on hierarchical grids (see [32] and [34]), which was used to preconditioning Steklov-Poincaré operators. The other one is the multipole method [29] (or $\mathcal{H}$ matrix method [14]), which is a completely approximate method for implementing the matrix-vector multiplication arising from boundary element methods. It is obvious that the new multilevel method is different from the multilevel method on hierarchical grids. The* *multipole method is based on boundary integral operators with singular kernels, and can not be implemented directly in parallel. The new multilevel method depends on only a suitable norm, and is a natural parallel algorithm. Besides, the new multilevel method has smaller computational complexity than the multipole method. For example, we consider the two-level multipole method for computing $V_{ij}\varphi_h$ for $\varphi_h \in W_h(\Gamma_{ij})$, which corresponds to the first level decomposition for Γ_{ij} (i.e., $J = 1$). It is known that the complexity for this computation is $O(n_{ij}^{\frac{3}{2}})$. But, only the complexity $O(n_{ij}^{\frac{4}{3}})$ is needed in the implementation of $\hat{M}_{ij}^{-1}(1)$.*

6 A variant of BPS preconditioner

In applications, the number N of the subdomains $\{\Omega_k\}$ would not large (otherwise, the coarser subspace $W_h^0(\Gamma)$ has high dimensions). This means that each subdomain Ω_k is not small, and so it is expensive to use exact solvers on Ω_k. In the rest of this paper, we study how to construct substructuring preconditioner with inexact solvers. For ease of understanding, we first investigate the direct way for such construction in this section.

6.1 Space decomposition

As usual, we define

$$V_h^p(\Omega_k) = \{v_h \in V_h\Omega : \ \textit{supp}\ v_h \subset \Omega_k\}.$$

Let $E_k : W_h(\partial\Omega_k) \to V_h(\Omega_k)$ be an inexact harmonic extension, which satisfies

$$|E_k\phi_h|_{1,\Omega_k} \lesssim |\phi_h|_{\frac{1}{2},\partial\Omega_k}, \quad \forall \phi_h \in W_h(\partial\Omega_k). \tag{6.1}$$

Define the approximate harmonic subspace

$$V_h^b(\Omega) = \{v_h \in V_h(\Omega) : \ v_h|_{\Omega_k} = E_k(\varphi_h|_{\partial\Omega_k}) \text{ for some } \varphi_h \in V_h(\Gamma)\}.$$

Then, we have the initial decomposition

$$V_h(\Omega) = \sum_{k=1}^{\mathrm{N}} V_h^p(\Omega_k) + V_h^b(\Omega).$$

Define the coarse subspace

$$V_h^0(\Omega) = \{v_h \in V_h^b(\Omega) : \ v_h|_{\Omega_k} = E_k(\varphi_h|_{\partial\Omega_k}) \ (k = 1, \cdots, N) \text{ for some } \varphi_h \in W_h^0(\Gamma)\}.$$

Define the local subspaces

$$V_h(\Omega_{\mathrm{E}}) = \{v_h \in V_h^b(\Omega) : \ v_h|_{\Omega_k} = E_k(\phi_h|_{\partial\Omega_k}) \ (k = 1, \cdots, \mathrm{N}), \text{ for some } \phi_h \in \tilde{W}_h(\mathrm{E})\},$$

and

$$V_h(\Omega_{ij}) = \{v_h \in V_h^b(\Omega) : \ v_h|_{\Omega_k} = E_k(\phi_h|_{\partial\Omega_k}) \ (k = 1, \cdots, \mathrm{N}), \text{ for some } \phi_h \in \tilde{W}_h(\Gamma_{ij})\}.$$

Then, we have

$$V_h(\Omega) = \sum_{k=1}^{\mathrm{N}} V_h^p(\Omega_k) + V_h^0(\Omega) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} V_h(\Omega_{\mathrm{E}}) + \sum_{\Gamma_{ij}} V_h(\Omega_{ij}).$$

6.2 Preconditioner

Let M_0, $M_{\rm E}$ and M_{ij} be the operators defined in the last section. Define
- the global coarse solver $B_0 : V_h^0(\Omega) \to V_h^0(\Omega)$:

$$(B_0 v_h, w_h) = \langle M_0(v_h|_\Gamma), w_h|_\Gamma\rangle_\Gamma, \quad v_h \in V_h^0(\Omega), \forall w_h \in V_h^0(\Omega); \tag{6.2}$$

- the edge solver $B_{\rm E} : V_h^0(\Omega) \to V_h^0(\Omega)$:

$$(B_{\rm E} v_h, w_h)_{\Omega_{\rm E}} = \langle M_{\rm E}(v_h|_{\rm E}), w_h|_{\rm E}\rangle_\Gamma, \quad v_h \in V_h(\Omega_{\rm E}), \forall w_h \in V_h(\Omega_{\rm E}); \tag{6.3}$$

- the face solver $B_{ij} : V_h(\Omega_{ij}) \to V_h(\Omega_{ij})$:

$$(B_{ij} v_h, w_h)_{\Omega_{ij}} = \langle M_{ij}(v_h|_{\Gamma_{ij}}), w_h|_{\Gamma_{ij}}\rangle_\Gamma, \quad v_h \in V_h(\Omega_{ij}), \forall w_h \in V_h(\Omega_{ij}). \tag{6.4}$$

Let $B_k : V_h^p(\Omega_k) \to V_h^p(\Omega_k)$ be a symmetric and positive definite operator satisfying

$$(B_k v_h, v_h)_{\Omega_k} \eqsim \int_{\Omega_k} \omega |\nabla v_h|^2 dp, \quad \forall v_h \in V_h^p(\Omega_k). \tag{6.5}$$

Note that the local solver B_k may be inexact.

Define the operator $A_h : V_h(\Omega) \to V_h(\Omega)$ by

$$(A_h v_h, w_h) = \mathcal{A}(v_h, w_h), \quad \forall v, \; w \in V_h(\Omega).$$

The preconditioner for A_h is defined by

$$B_h^{-1} = \sum_{k=1}^{\rm N} B_k^{-1} Q_k^* + B_0^{-1} Q_0^* + \sum_{{\rm E} \in \mathcal{E}_\Gamma} B_{\rm E}^{-1} Q_{\rm E}^* + \sum_{\Gamma_{ij} \subset \Gamma} B_{ij}^{-1} Q_{ij}^*, \tag{6.6}$$

where $Q_k^* : V_h(\Omega) \to V_h^0(\Omega_k)$, $Q_0^* : V_h(\Omega) \to V_h^0(\Omega)$, $Q_{\rm E}^* : V_h(\Omega) \to V_h(\Omega_{\rm E})$ and $Q_{ij}^* : V_h(\Omega) \to V_h(\Omega_{ij})$ denote the L^2 projections.

Theorem 6.1 *For the preconditioner B_h, we have*

$$cond(B_h^{-1} A_h) \le C[1 + \log(H/h)]^2, \tag{6.7}$$

where C is a constant independent of h, H and the jumps of the coefficient ω across the faces Γ_{ij}.

6.3 Analysis

By the convergence theory (see [31], [35] and [40]), one needs to verify the condition: For any $v_h \in V_h(\Omega)$, there exists a decomposition

$$v_h = \sum_{k=1}^{\rm N} v_k + v_0 + \sum_{{\rm E} \in \mathcal{E}_\Gamma} v_{\rm E} + \sum_{\Gamma_{ij}} v_{ij}$$

with

$$v_k \in V_h^p(\Omega_k), \quad v_0 \in V_h^0(\Omega), \quad v_{\rm E} \in V_h(\Omega_{\rm E}) \;\text{ and }\; v_{ij} \in V_{ij},$$

such that

$$\sum_{k=1}^{\rm N} (B_k v_k, B_k)_{\Omega_k} + (B_0 v_0, v_0) + \sum_{{\rm E} \in \mathcal{E}_\Gamma} (B_{\rm E} v_{\rm E}, v_{\rm E})_{\Omega_{\rm E}} + \sum_{\Gamma_{ij}} (B_{ij} v_{ij}, v_{ij})_{\Omega_{ij}}$$

$$\lesssim [1+\log(H/h)]^2(A_h v_h, v_h). \tag{6.8}$$

Set $\phi_h = v_h|_\Gamma$, and let $\phi_0 \in W_h^0(\Gamma)$, $\phi_{\rm E} \in \tilde{W}_h({\rm E})$ and $\phi_{ij} \in \tilde{W}_h(\Gamma_{ij})$. Define v_0, $v_{\rm E}$ and v_{ij} by

$$v_0|_{\Omega_k} = E_k(\phi_0|_{\partial\Omega_k}), \quad v_{\rm E}|_{\Omega_k} = E_k(\phi_{\rm E}|_{\partial\Omega_k}) \text{ and } v_{ij}|_{\Omega_k} = E_k(\varphi_{ij}|_{\partial\Omega_k}) \quad (k=1,\cdots,{\rm N}).$$

Then,

$$v_0 \in V_h^0(\Omega), \quad v_{\rm E} \in V_h(\Omega_{\rm E}) \text{ and } v_{ij} \in V_h(\Omega_{ij}).$$

Set

$$v_k = [v_h - (v_0 + \sum_{{\rm E}\in\mathcal{E}_\Gamma} v_{\rm E} + \sum_{\Gamma_{ij}} v_{ij})]|_{\Omega_k}, \quad k=1,\cdots,{\rm N}. \tag{6.9}$$

It is easy to see that $v_k \in V_h^p(\Omega_k)$ for each k. Moreover, we have

$$v_h = \sum_{k=1}^{\rm N} v_k + v_0 + \sum_{{\rm E}\in\mathcal{E}_\Gamma} v_{\rm E} + \sum_{\Gamma_{ij}} v_{ij}.$$

It is clear that

$$(v_0 + \sum_{{\rm E}\in\mathcal{E}_\Gamma} v_{\rm E} + \sum_{\Gamma_{ij}} v_{ij})|_{\Omega_k} = E_k[(\phi_0 + \sum_{{\rm E}\in\mathcal{E}_\Gamma} \phi_{\rm E} + \sum_{\Gamma_{ij}} \phi_{ij})|_{\partial\Omega_k}].$$

Thus, we get by (6.5) and (6.9)

$$\begin{aligned}(B_k v_k, v_k)_{\Omega_k} &\lesssim \int_{\Omega_k} \omega|\nabla v_k|^2 dp \\ &\lesssim \omega_k(|v_h|^2_{1,\Omega_k} + |E_k[(\phi_0 + \sum_{{\rm E}\in\mathcal{E}_\Gamma} \phi_{\rm E} + \sum_{\Gamma_{ij}} \phi_{ij})|_{\partial\Omega_k}]|^2_{1,\Omega_k}).\end{aligned}$$

This, together with (6.1), leads to

$$(B_k v_k, v_k) \lesssim \omega_k(|v_h|^2_{1,\Omega_k} + |\phi_0 + \sum_{{\rm E}\in\mathcal{E}_\Gamma} \phi_{\rm E} + \sum_{\Gamma_{ij}} \phi_{ij}|_{\frac{1}{2},\partial\Omega_k}). \tag{6.10}$$

By (3.12) and the trace theorem, we have

$$|\phi_0 + \sum_{{\rm E}\in\mathcal{E}_\Gamma} \phi_{\rm E} + \sum_{\Gamma_{ij}} \phi_{ij}|^2_{\frac{1}{2},\partial\Omega_k} = |\varphi_h|^2_{\frac{1}{2},\partial\Omega_k} \lesssim |v_h|^2_{1,\Omega_k}.$$

Plugging this in (6.10), leads to

$$(B_k v_k, v_k)_{\Omega_k} \lesssim \omega_k |v_h|^2_{1,\Omega_k}, \quad k=1,\cdots,{\rm N}.$$

Thus,

$$\sum_{k=1}^{\rm N} (B_k v_k, v_k)_{\Omega_k} \lesssim (A_h v_h, v_h). \tag{6.11}$$

On the other hand, we have by (6.2) and (5.1)

$$(B_0 v_0, v_0) \eqsim \|\phi_0\|^2_{*,\Gamma}. \tag{6.12}$$

Similarly, it follows, by (6.4) and (5.2), that

$$(B_{ij} v_{ij}, v_{ij})_{\Omega_{ij}} \eqsim \|\phi_{ij}\|^2_{*,\Gamma}. \tag{6.13}$$

Moreover, we have from the definition of $M_{\rm E}$

$$\langle M_{\rm E}\phi_{\rm E}, \phi_{\rm E}\rangle \eqsim \|\phi_{\rm E}\|^2_{*,\Gamma}.$$

Substituting this into (6.3), yields

$$(B_{\rm E}v_{\rm E}, v_{\rm E})_{\Omega_{\rm E}} \eqsim \|\phi_{\rm E}\|^2_{*,\Gamma}. \tag{6.14}$$

Combining (6.12)-(6.14), and using (3.13), yields

$$\begin{aligned}(B_0v_0, v_0) + \sum_{{\rm E}\in\mathcal{E}_\Gamma}(B_{\rm E}v_{\rm E}, v_{\rm E}) + \sum_{\Gamma_{ij}}(B_{ij}v_{ij}, v_{ij}) &\lesssim [1+\log(H/h)]^2\|\phi_h\|^2_{*,\Gamma}\\ &\lesssim [1+\log(H/h)]^2(A_hv_h, v_h).\end{aligned}$$

This, together with (6.11), gives (6.8).
□

Remark 6.1 *When each subdomain solver B_k is exact, i.e.,*

$$(B_kv_h, w_h)_{\Omega_k} = \omega_k(\nabla v_h, \nabla w_h)_{\Omega_k}, \quad v_h \in V_h^p(\Omega_k),\ \forall w_h \in V_h^p(\Omega_k),$$

the inequality (6.11) can be obtained directly by the orthogonality (see [5] and [40]). But, if some B_k is inexact, we have to derive (6.11) as above.

6.4 Implementation and problem

Let the stiffness matrices of M_0, $M_{\rm E}^{-1}$ and M_{ij}^{-1} be computed as in Section 4.

We first describe a direct algorithm to implement the action of B_h^{-1}.

Algorithm 6.1. For a $g \in V_h(\Omega)$, the solution $u_g \in V_h(\Omega)$ satisfying

$$(B_hu_g, v_h) = (g, v_h), \quad \forall v_h \in V_h(\Omega)$$

can be gotten as follows:

Step 1. Computing $u_k^p \in V_k^p(\Omega_k)$ in parallel by

$$(B_ku^k, v_h)_{\Omega_k} = (g, v_h)_{\Omega_k}, \quad \forall v_h \in V_k^p(\Omega_k);$$

Step 2. Computing $\phi_0 \in W_h^0(\Gamma)$ by

$$\langle M_0\phi_0, \psi_h\rangle_\Gamma = (g, v_h), \quad v_h|_{\Omega_k} = E_k(\psi_h|_{\partial\Omega_k}), \quad \forall\psi_h \in W_h^0(\Gamma);$$

Step 3. Computing $\phi_{\rm E} \in \tilde{W}_h({\rm E})$ in parallel by

$$\langle M_{\rm E}\phi_{\rm E}, \psi_h\rangle_{\rm E} = (g, v_h)_{\Omega_{\rm E}}, \quad v_h|_{\Omega_k} = E_k(\psi_h|_{\partial\Omega_k}), \quad \forall\psi_h \in \tilde{W}_h({\rm E});$$

Step 4. Computing $\phi_{ij} \in \tilde{W}_h(\Gamma_{ij})$ in parallel by

$$\langle M_{ij}\phi_{ij}, \psi_h\rangle_{\Gamma_{ij}} = (g, v_h)_{\Omega_{ij}}, \quad v_h|_{\Omega_k} = E_k(\psi_h|_{\partial\Omega_k}), \quad \forall\psi_h \in \tilde{W}_h(\Gamma_{ij});$$

Step 5. For $\phi_h = \phi_0 + \sum\limits_{{\rm E}\in\mathcal{E}_\Gamma}\phi_{\rm E} + \sum\limits_{\Gamma_{ij}}\phi_{ij}$, compute $E_k(\phi_h|_{\partial\Omega_k})$, and define u_g by $u_g|_{\Omega_k} = u_k^p + E_k(\phi_h|_{\partial\Omega_k})$.

Remark 6.2 *It is very expensive to implement Algorithm 6.1 directly. The main difficulty for implementing Algorithm 6.1 is that we have to compute the (approximate) harmonic extensions* $E_k(\psi_h|_{\partial\Omega_k})$ $(k = 1, \cdots, \mathrm{N})$ *for each basis function* ψ_h *of* $W_h^0(\Gamma)$, $\tilde{W}_h(\mathrm{E})$ *and of* $\tilde{W}_h(\Gamma_{ij})$. *It is clear that the dimensions of* $V_h(\partial\Omega_k)$ *equal to* $O((H/h)^2)$. *Thus, the complexity for computing these harmonic extensions is* $O(\mathrm{N} \times (H/h)^5)$ *at least, even if the complexity for computing such a harmonic extension equals to* $O((H/h)^3)$, *which is optimal.*

Let each B_k be exact. Then, Algorithm 6.1 can be described in a different manner (refer to [5] and [40]). In the following algorithm, let $\tilde{\psi}_h \in V_h(\Omega)$ denote the zero extension of $\psi_h \in W_h(\Gamma)$.
Algorithm 6.2 (with exact solvers B_k). Let each B_k is exact. For a $g \in V_h(\Omega)$, the solution $u_g \in V_h(\Omega)$ satisfying

$$(B_h u_g, v_h) = (g, v_h), \quad \forall v_h \in V_h(\Omega)$$

can be gotten as follows :

Step 1. Computing $u_k^p \in V_k^p(\Omega_k)$ in parallel by

$$(B_k u_k^p, v_h)_{\Omega_k} = (g, v_h)_{\Omega_k}, \quad \forall v_h \in V_k^p(\Omega_k);$$

Step 2′. Computing $\phi_0 \in W_h^0(\Gamma)$ by

$$\langle M_0\phi_0, \psi_h\rangle_\Gamma = (g, \tilde{\psi}_h) - \sum_{k=1}^{\mathrm{N}} \omega_k(\nabla u_k, \nabla\tilde{\psi}_h)_{\Omega_k}, \quad \forall \psi_h \in W_h^0(\Gamma);$$

Step 3′. Computing $\phi_\mathrm{E} \in \tilde{W}_h(\mathrm{E})$ in parallel by

$$\langle M_\mathrm{E}\phi_\mathrm{E}, \psi_h\rangle_\mathrm{E} = (g, \tilde{\psi}_h^0)_{\Omega_\mathrm{E}} - \sum_{k\in\mathcal{Q}_\mathrm{E}} \omega_k(\nabla u_k^p, \nabla\tilde{\psi}_h^0)_{\Omega_k}, \quad \forall \psi_h \in \tilde{W}_h(\mathrm{E});$$

Step 4′. Computing $\phi_{ij} \in \tilde{W}_h(\Gamma_{ij})$ in parallel by

$$\langle M_{ij}\phi_{ij}, \psi_h\rangle_{\Gamma_{ij}} = (g, \tilde{\psi}_h)_{\Omega_{ij}} - \sum_{k=i,j} \omega_k(\nabla u_k, \nabla\tilde{\psi}_h)_{\Omega_k}, \quad \forall \psi_h \in \tilde{W}_h(\Gamma_{ij});$$

Step 5. For $\phi_h = \phi_0 + \sum\limits_{\mathrm{E}\in\mathcal{E}_\Gamma} \phi_\mathrm{E} + \sum\limits_{\Gamma_{ij}} \phi_{ij}$, compute $E_k(\phi_h|_{\partial\Omega_k})$, and define u_g by $u_g|_{\Omega_k} = u_k^p + E_k(\phi_h|_{\partial\Omega_k})$.

It is clear that Step 2′ is much cheaper than Step 2 in [5]. Like the most existing substructuring methods, it is in general expensive to implement Step 1 in Algorithm 6.2, except that each subspace $V_h^p(\Omega_k)$ possesses low dimensions. But, the space $V_h^p(\Omega_k)$ has high dimensions in most applications (otherwise, the coarser space $W_h^0(\Gamma)$ is large).

In the next two sections, we propose new methods to solve the problems mentioned above and in Remark 6.2 by using the multilevel technique developed in Section 4 and Section 5.

7 Substructuring method with multilevel solvers

7.1 Multilevel decomposition for $V_h(\Omega)$

Let $E_0 : W_h^0(\Gamma) \to V_h(\Omega)$ and $E_\mathrm{E} : \tilde{W}_h(\mathrm{E}) \to V_h(\Omega_\mathrm{E})$ be (approximate) harmonic extensions satisfying

$$\sum_{k=1}^{\mathrm{N}} \omega_k |E_0\phi_0|_{1,\Omega_k}^2 \lesssim \|\phi_0\|_{*,\Gamma}^2, \quad \forall \phi_0 \in W_h^0(\Gamma), \tag{7.1}$$

and

$$\sum_{k\in\mathcal{Q}_{\rm E}}\omega_k|E_{\rm E}\phi_{\rm E}|^2_{1,\Omega_k}\lesssim\|\phi_{\rm E}\|^2_{*,\Gamma},\quad\forall\phi_{\rm E}\in\tilde{W}_h({\rm E}).\tag{7.2}$$

For each $\Gamma^{(k)}_{ij,\ r}$ $(k=0,\cdots,J;\ r=1,\cdots,{\rm M}_k)$, choose an auxiliary polyhedron $\Omega^{(k)}_{ij,\ r}$, which is a subdomain of Ω_{ij}, and satisfies $\Omega^{(k)}_{ij,\ r}\cap\Gamma_{ij}=\Gamma^{(k)}_{ij,\ r}$. Moreover, we require that $\Omega^{(k)}_{ij,\ r}$ has almost the same size d_k with $\Gamma^{(k)}_{ij,\ r}$. For example, for $\Gamma^{(0)}_{ij,\ 1}=\Gamma_{ij}$, one can choose $\Omega^{(0)}_{ij,\ 1}$ to be Ω_{ij} itself. Let $\hat{V}_h(\Omega^{(k)}_{ij,\ r})$ be a suitable subspace of $V_h(\Omega_{ij})$, such that each function in $\hat{V}_h(\Omega^{(k)}_{ij,\ r})$ vanishes outside $\Omega^{(k)}_{ij,\ r}$. Design a suitable harmonic extensions $E^{(k)}_{ij,\ r,\ 0}:W^0_h(\Gamma^{(k)}_{ij,\ r})\rightarrow\hat{V}_h(\Omega^{k)}_{ij,\ r})$ $(k=1,\cdots,J-1)$ and $E^{J)}_{ij,\ r}:\tilde{W}_h(\bar{\Gamma}^{(J)}_{ij,\ r})\rightarrow\hat{V}_h(\Omega^{(J)}_{ij,\ r})$, such that

$$\begin{aligned}&\omega_i|E^{(k)}_{ij,\ r,\ 0}\varphi_h|^2_{1,\Omega_i}+\omega_j|E^{(k)}_{ij,\ r,\ 0}\varphi_h|^2_{1,\Omega_j}\\&\lesssim\|\varphi_h\|^2_{*,\Gamma},\quad\forall\varphi_h\in W^0_h(\Gamma^{(k)}_{ij,\ r})\ \ (k=1,\cdots,J-1),\end{aligned}\tag{7.3}$$

and

$$\omega_i|E^{(J)}_{ij,\ r}\varphi_h|^2_{1,\Omega_i}+\omega_j|E^{(J)}_{ij,\ r}\varphi_h|^2_{1,\Omega_j}\lesssim\|\phi_h\|^2_{*,\Gamma},\quad\forall\varphi_h\in\tilde{W}_h(\bar{\Gamma}^{(J)}_{ij,\ r}).\tag{7.4}$$

Define the subspaces

$$V^0_h(\Omega^{(k)}_{ij,\ r})=\{v_h\in V_h(\Omega_{ij}):\ v_h=E^{(k)}_{ij,\ r,\ 0}\phi_h\ \text{for some}\ \phi_h\in W^0_h(\Gamma^{(k)}_{ij,\ r})\}\ \ (k=1,\cdots,J-1),$$

and

$$\tilde{V}_h(\Omega^{(J)}_{ij,\ r})=\{v_h\in V_h(\Omega_{ij}):\ v_h=E^{(J)}_{ij,\ r}\phi_h\ \text{for some}\ \phi_h\in\tilde{W}_h(\bar{\Gamma}_{ij,\ r})\}.$$

Since we have

$$W_h(\Gamma)=W^0_h(\Gamma)+\sum_{{\rm E}\in\mathcal{E}_\Gamma}\tilde{W}_h({\rm E})+\sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{{\rm M}_k}W^0_h(\Gamma^{(k)}_{ij,\ r})+\sum_{k=1}^{{\rm M}_J}\tilde{W}_h(\bar{\Gamma}^{(J)}_{ij,\ r})],$$

the space decomposition holds

$$V_h(\Omega)=V^0_h(\Omega)+\sum_{{\rm E}\in\mathcal{E}_\Gamma}V_h(\Omega_{\rm E})+\sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{{\rm M}_k}V^0_h(\Omega^{(k)}_{ij,\ r})\sum_{r=1}^{{\rm M}_J}\tilde{V}_h(\Omega^{(J)}_{ij,\ r})].$$

7.2 New substructuring preconditioner

Define B_k, B_0 and $B_{\rm E}$ so that (6.5), (6.2) and (6.3) hold. Let $M^{(k)}_{ij,\ r,\ 0}$ $(k=0,\cdots,J-1)$ and $M^{(J)}_{ij,\ r}$ be defined as in Section 5. Define

- local coarse solvers $B^{(k)}_{ij,\ r,\ 0}:V^0_h(\Omega^{(k)}_{ij,\ r})\rightarrow V^0_h(\Omega^{(k)}_{ij,\ r})$ $(k=0,\cdots,J-1)$ by

$$(B^{(k)}_{ij,\ r,\ 0}v_h,w_h)=\langle M^{(k)}_{ij,\ r,\ 0}(v_h|_{\Gamma^{(k)}_{ij,\ r}}),w_h|_{\Gamma^{(k)}_{ij,\ r}}\rangle_{\Gamma^{(k)}_{ij,\ r}},\quad v_h\in V^0_h(\Omega^{(k)}_{ij,\ r}),\ \forall w_h\in V^0_h(\Omega^{(k)}_{ij,\ r});$$

- the finest local solvers $\tilde{B}^{(J)}_{ij,\ r}:\tilde{V}_h(\Omega^{(J)}_{ij,\ r})\rightarrow\tilde{V}_h(\Omega^{(J)}_{ij,\ r})$ by

$$(B^{(J)}_{ij,\ r}v_h,w_h)=\langle M^{(J)}_{ij,\ r}(v_h|_{\Gamma^{(J)}_{ij,\ r}}),w_h|_{\Gamma^{(J)}_{ij,\ r}}\rangle_{\Gamma^{(J)}_{ij,\ r}},\quad v_h\in\tilde{V}_h(\Omega^{(J)}_{ij,\ r}),\ \forall w_h\in\tilde{V}_h(\Omega^{(J)}_{ij,\ r}).$$

From Section 5, we know that these solvers satisfy

$$(B^{(k)}_{ij,\ r,\ 0}v_h,v_h)\eqsim(\omega_i+\omega_j)|v_h|^2_{H^{\frac{1}{2}}_{00}(\Gamma^{(k)}_{ij,\ r})},\quad\forall v_h\in V^0_h(\Omega^{(k)}_{ij,\ r}),\tag{7.5}$$

and

$$(\tilde{B}^{(J)}_{ij,\ r}v_h, v_h) \eqsim (\omega_i+\omega_j)|v_h|^2_{H^{\frac{1}{2}}_{00}(\Gamma^{(J)}_{ij,\ r})}, \quad \forall v_h \in \tilde{V}_h(\Omega^{(J)}_{ij,\ r}). \tag{7.6}$$

Now, a multilevel preconditioner for A_h is defined as

$$\begin{aligned} B_J^{-1} &= \sum_{k=1}^{\mathrm{N}} B_k^{-1}Q_k^* + B_0^{-1}Q_0^* + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} B_{\mathrm{E}}^{-1}Q_{\mathrm{E}}^* \\ &+ \sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}(B^{(k)}_{ij,\ r,\ 0})^{-1}Q^{(k),*}_{ij,\ r,\ 0} + \sum_{r=1}^{\mathrm{M}_J}(\tilde{B}^{(J)}_{ij,\ r})^{-1}Q^{(J),*}_{ij,\ r}], \end{aligned}$$

where $Q^{(k),*}_{ij,\ r,\ 0} : L^2(\Omega) \to V_h^0(\Omega^{(k)}_{ij,\ r})$ $(k=0,\cdots,J-1)$, and $Q^{(J),*}_{ij,\ r} : L^2(\Omega) \to \tilde{V}_h(\Omega^{(J)}_{ij,\ r})$ denote L^2 projectors.

Theorem 7.1 *For the preconditioner B_J, we have*

$$cond(B_J^{-1}A_h) \leq CJ^3[1+\log(H/h)]^2, \tag{7.7}$$

where C is a constant independent of h, H, d_k and the jumps of the coefficient ω across the faces Γ_{ij}.

7.3 Analysis

One needs to establish a suitable decomposition for $v_h \in V_h(\Omega)$

$$v_h = \sum_{k=1}^{\mathrm{N}} v_k + v_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} v_{\mathrm{E}} + \sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l} v^{(l)}_{ij,\ r\ ,0} + \sum_{r=1}^{\mathrm{M}_J}\tilde{v}^{(J)}_{ij,\ r}], \tag{7.8}$$

with

$$v_k \in V_h^p(\Omega_k) \quad (k=1,\cdots,\mathrm{N}), \quad v_0 \in V_h^0(\Omega), \quad v_{\mathrm{E}} \in V_h(\Omega_{\mathrm{E}}),$$

and

$$v^{(l)}_{ij,\ r,\ 0} \in V_h^0(\Omega^{(l)}_{ij,\ r}) \quad (l=0,\cdots,J-1) \quad \text{and} \quad v^{(J)}_{ij,\ r} \in \tilde{V}_h(\Omega^{(J)}_{ij,\ r}.$$

This decomposition should satisfy the stability condition

$$\begin{aligned} &\sum_{k=1}^{\mathrm{N}}(B_kv_k,\ v_k)_{\Omega_k} + (B_0v_0,\ v_0) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma}(B_{\mathrm{E}}v_{\mathrm{E}},\ v_{\mathrm{E}})_{\Omega_{\mathrm{E}}} \\ &+\sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}(B^{(l)}_{ij,\ r,\ 0}v^{(l)}_{ij,\ r,\ 0},\ v^{(l)}_{ij,\ r,\ 0}) + \sum_{r=1}^{\mathrm{M}_J}(B^{(J)}_{ij,\ r}\tilde{v}^{(J)}_{ij,\ r}, \tilde{v}^{(J)}_{ij,\ r})] \\ &\lesssim J^2[1+\log(H/h)]^2(A_hv_h, v_h). \end{aligned} \tag{7.9}$$

Set $\phi_h = v_h|_\Gamma$, and decompose ϕ_h into

$$\phi_h = \phi_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma}\phi_{\mathrm{E}} + \sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}\phi^{(l)}_{ij,\ r,\ 0} + \sum_{r=1}^{\mathrm{M}_J}\tilde{\phi}^{(J)}_{ij,\ r}]. \tag{7.10}$$

Hereafter, ϕ_0, ϕ_{E}, $\phi^{(l)}_{ij,\ r,\ 0}$ $(l=0,\cdots,J-1)$ and $\tilde{\phi}^{(J)}_{ij,\ r}$ are defined by Section 4. Define

$v_0 = E_0\phi_0$, $v_{\mathrm{E}} = E_{\mathrm{E}}\phi_{\mathrm{E}}$, $v^{(l)}_{ij,\ r,\ 0} = E^{(l)}_{ij,\ r,\ 0}\phi^{(l)}_{ij,\ r,\ 0}$ $(l=0,\cdots,J-1)$ and $\tilde{v}^{(J)}_{ij,\ r} = E^{(J)}_{ij,\ r}\tilde{\phi}^{(J)}_{ij,\ r}$.

Moreover, we set for each Ω_k

$$v_k = \{v_h - v_0 - \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} v_{\mathrm{E}} - \sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l} v_{ij,\ r\ ,0}^{(l)} + \sum_{r=1}^{\mathrm{M}_J}\tilde{v}_{ij,\ r}^{(J)}]\}|_{\Omega_k}. \tag{7.11}$$

Then, we have $v_k \in V_h^p(\Omega_k)$ by (7.10). It suffices to verify (7.9) for the functions defined above.

For convenience, set

$$\begin{aligned}\mathcal{G}(\phi_h) &= (B_0v_0, v_0) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma}(B_{\mathrm{E}}v_{\mathrm{E}},\ v_{\mathrm{E}})_{\Omega_{\mathrm{E}}} \\ &+ \sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}(B_{ij,\ r,\ 0}^{(l)}v_{ij,\ r,\ 0}^{(l)},\ v_{ij,\ r,\ 0}^{(l)}) + \sum_{r=1}^{\mathrm{M}_J}(B_{ij,\ r}^{(J)}\tilde{v}_{ij,\ r}^{(J)}, \tilde{v}_{ij,\ r}^{(J)})].\end{aligned}$$

By (6.2), (6.3) and (7.5)-(7.6), we deduce

$$\begin{aligned}\mathcal{G}(\phi_h) &\lesssim \|v_0\|_{*,\Gamma}^2 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma}\|v_{\mathrm{E}}\|_{*,\Gamma}^2 \\ &+ \sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}\|v_{ij,\ r,\ 0}^{(l)}\|_{*,\ \Gamma}^2 + \sum_{r=1}^{\mathrm{M}_J}\|\tilde{v}_{ij,\ r}^{(J)}\|_{*,\ \Gamma}^2].\end{aligned} \tag{7.12}$$

Here, we have used the fact that

$$|\phi_h|_{H_{00}^{\frac{1}{2}}(\Gamma_{ij,\ r}^{(l)})} \eqsim |\phi_h|_{\frac{1}{2},\partial\Omega_i} \eqsim |\phi_h|_{\frac{1}{2},\partial\Omega_j}, \quad \forall\phi_h \in H_{00}^{\frac{1}{2}}(\Gamma_{ij,\ r}^{(l)})\ (l = 0,\cdots,J;\ r = 1,\cdots,\mathrm{M}_l).$$

Combining (4.9) and (7.12), and using the trace theorem, yields

$$\mathcal{G}(\phi_h) \lesssim J[1+\log(H/h)]^2\|\phi_h\|_{*,\Gamma}^2 \lesssim J[1+\log(H/h)]^2(A_hv_h, v_h). \tag{7.13}$$

Besides, it follows by (7.11) that

$$\begin{aligned}|v_k|_{1,\Omega_k}^2 &\lesssim |v_h|_{1,\Omega_k}^2 + |v_0|_{1,\Omega_k}^2 + |\sum_{\mathrm{E}\in\mathcal{E}_\Gamma} v_{\mathrm{E}}|_{1,\Omega_k}^2 \\ &+ J\sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}|v_{ij,\ r,\ 0}^{(l)}|_{1,\ \Omega_k}^2 + \sum_{r=1}^{\mathrm{M}_J}|\tilde{v}_{ij,\ r}^{(J)}|_{1,\ \Omega_k}^2].\end{aligned}$$

This, together with (6.5), (6.2), (6.3) and (7.5)-(7.6), leads to

$$\sum_{k=1}^{\mathrm{N}}(B_kv_k, v_k)_{\Omega_k} \lesssim (A_hv_h, v_h) + J\mathcal{G}(\phi_h).$$

Now, (7.9) is a direct consequence of the above inequality, together with (7.13).

On the other hand, for each subspace $V_h^0(\Omega_{ij,\ r_0}^{(l_0)})$, there exists J subspaces among $V_h^0(\Omega_{ij,\ r}^{(l)})$ $(l = 0,\cdots,J-1;\ r = 1,\cdots,\mathrm{M}_l)$, which are not orthogonal with $V_h^0(\Omega_{ij,\ r_0}^{(l_0)})$. Thus, we deduce (8.3) by the convergence theory [38], together with (7.9).

□

Remark 7.1 *A direct idea is to consider the multilevel extension operator* $E_h : W_h(\Gamma) \to V_h(\Omega)$ *defined by*

$$E_h\phi_h = E_0\phi_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} E_{\mathrm{E}}\phi_{\mathrm{E}}$$

$$+ \quad \sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k} E^{(k)}_{ij,\ r,\ 0}\phi^{(k)}_{ij,\ r,\ 0} + \sum_{r=1}^{\mathrm{M}_J} E^{(J)}_{ij,\ r}\tilde{\phi}^{(J)}_{ij,\ r},$$

and define B_J as in B_h (see Section 6.1). But, there are some difficulties in the implementation of such B_J^{-1}.

7.4 Implementation

Let B_0, B_{E}, $B^{(k)}_{ij,\ r,\ 0}$ $(k = 0, \cdots, J-1)$ and $B^{(J)}_{ij,\ r}$ be defined as in Subsection 6.2 and Subsection 7.2. The action of B_J^{-1} can be described by the following algorithm

Algorithm 7.1. For a $g \in V_h(\Omega)$, the solution $u_g \in V_h(\Omega)$ satisfying

$$(B_J u_g, v_h) = (g, v_h), \quad \forall v_h \in V_h(\Omega)$$

can be gotten as follows:

Step 1. Computing $u^p_k \in V^p_k(\Omega_k)$ in parallel by

$$(B_k u^p_k, v_h)_{\Omega_k} = (g, v_h)_{\Omega_k}, \quad \forall v_h \in V^p_k(\Omega_k);$$

Step 2. Computing $\phi_0 \in W^0_h(\Gamma)$ by

$$\langle M_0\phi_0, \psi_h\rangle_\Gamma = (g, v_h), \quad v_h = E_0\psi_h, \quad \forall \psi_h \in W^0_h(\Gamma);$$

Step 3. Computing $\phi_{\mathrm{E}} \in \tilde{W}_h(\mathrm{E})$ in parallel by

$$\langle M_{\mathrm{E}}\phi_{\mathrm{E}}, \psi_h\rangle_{\mathrm{E}} = (g, v_h)_{\Omega_{\mathrm{E}}}, \quad v_h = E_{\mathrm{E}}\psi_h, \quad \forall \psi_h \in \tilde{W}_h(\mathrm{E});$$

Step 4. Computing $\phi^{(l)}_{ij,\ r,\ 0} \in W^0_h(\Gamma^{(l)}_{ij,\ r})$ in parallel by

$$\langle M^{(l)}_{ij,\ r,\ 0}\phi^{(l)}_{ij,\ r,\ 0},\ \psi_h\rangle_{\Gamma^{(l)}_{ij,\ r}} = (g, v_h)_{\Omega^{(l)}_{ij,\ r,\ 0}}, \quad v_h = E^{(l)}_{ij,\ r,\ 0}\psi_h, \quad \forall \psi_h \in W^0_h(\Gamma^{(l)}_{ij,\ r});$$

Step 5. Computing $\tilde{\phi}^{(J)}_{ij,\ r} \in \tilde{W}_h(\Gamma^{(J)}_{ij,\ r})$ in parallel by

$$\langle M^{(J)}_{ij,\ r}\tilde{\phi}^{(J)}_{ij,\ r}, \psi_h\rangle_{\Gamma^{(J)}_{ij,\ r}} = (g, v_h)_{\Omega^{(J)}_{ij,\ r}}, \quad v_h = E^{(J)}_{ij,\ r}\psi_h, \quad \forall \psi_h \in \tilde{W}_h(\Gamma^{(J)}_{ij,\ r});$$

Step 6. Computing

$$v_0 = E_0\phi_0, \quad v_{\mathrm{E}} = E_{\mathrm{E}}\phi_{\mathrm{E}}, \quad v^{(l)}_{ij,\ r,\ 0} = E^{(l)}_{ij,\ r,\ 0}\phi^{(l)}_{ij,\ r,\ 0} \quad (l = 0, \cdots, J-1)$$

and

$$\tilde{v}^{(J)}_{ij,\ r} = E^{(J)}_{ij,\ r}\phi^{(J)}_{ij,\ r}.$$

Step 7. Set

$$u_g \quad = \quad \sum_{k=1}^{\mathrm{N}} v^p_k + v_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} v_{\mathrm{E}} + \sum_{\Gamma_{ij}}[\sum_{l=1}^{J-1}\sum_{r=1}^{\mathrm{M}_k} v^{(l)}_{ij,\ r,\ 0} + \sum_{r=1}^{\mathrm{M}_J}\tilde{v}^{(J)}_{ij,\ r}].$$

Before implementing Algorithm 7.1, one needs to compute the (approximate) harmonic basis $E_0\varphi_h$, $E_{\mathrm{E}}\varphi_h$, $E^{(k)}_{ij,\ r,\ 0}\varphi_h$ and $E^{(J)}_{ij,\ r}\varphi_h$ for each basis function φ_h of $W^0_h(\Gamma)$, $\tilde{W}_h(\mathrm{E})$, $W^0_h(\Gamma^{(k)}_{ij,\ r,\ 0})$ or of $\tilde{W}_h(\Gamma^{(J)}_{ij,\ r})$. After get such basis, which depends on approximate harmonic extensions, there is almost no cost in Step 6 and Step 7. In the next section, we develop a kind of approximate harmonic extension, so that the cost for computing each extension is optimal. Now, we first investigate the total computational complexity under this condition.

Let $N(J)$ denote the total complexity for computing all the harmonic basis functions.

Proposition 7.1. Let $m \geq 2$ be a given positive integer. Choose J and m_k $(k = 1, \cdots, J)$ by

$$J \eqsim \log_{\frac{3}{2}}(\log_m(H/h) + \frac{2}{3}), \quad m_1 = m, \quad \text{and} \quad m_k \eqsim m^{(\frac{3}{2})^{k-1}} \quad (k = 2, \cdots, J), \tag{7.14}$$

respectively. Assume that the cost for computing each local harmonic extension is optimal. Then,

$$N(J) \lesssim \log_{\frac{3}{2}}(\log_m(H/h)) \times [\mathrm{N} \times (H/h)^3], \tag{7.15}$$

which is almost optimal.

Proof. Since the dimensions of the global coarse space $W_h^0(\Gamma)$ equal to $O(\mathrm{N})$, the cost for computing $E_0\phi_h$ with all basis functions of $W_h^0(\Gamma)$ equals to

$$N_0 = O(\mathrm{N} \times (H/h)^3). \tag{7.16}$$

One can define E_{E} to be the simple zero extension, which satisfies (7.2). Then, the cost for computing $E_{\mathrm{E}}\phi_h$ with all basis functions of each $\tilde{W}_h(\mathrm{E})$ is

$$N_{\mathrm{E}} = O(H/h).$$

Note that the number of the coarse edges $\mathrm{E} \in \mathcal{E}_\Gamma$ equals to $O(\mathrm{N})$, we have

$$\sum_{\mathrm{E} \in \mathcal{E}_\Gamma} N_{\mathrm{E}} = O(\mathrm{N} \times H/h). \tag{7.17}$$

Consider a face Γ_{ij}. It is easy to see that the number of nodes in $\Gamma_{ij,\ r}^{(k)}$ is about n_{ij}/M_k, and so the number of nodes in $\Omega_{ij,\ r}^{(k)}$ is about $(n_{ij}/\mathrm{M}_k)^{\frac{3}{2}}$ with $\mathrm{M}_0 = 1$ $(k = 0, \cdots, J)$. By the definition of $W_h^0(\Gamma_{ij,\ r}^{(k)})$, the dimensions of each $W_h^0(\Gamma_{ij,\ r}^{(k)})$ for a k equal to $O(m_{k+1})$ $(k = 0, \cdots, J-1)$. Moreover, the number of the local coarse subspaces $W_h^0(\Gamma_{ij,\ r}^{(k)})$ for a k equals to M_k $(k = 0, \cdots, J-1)$. Besides, the dimensions of each $\tilde{W}_h(\bar{\Gamma}_{ij,\ r}^{(J)})$ equals to n_{ij}/M_J. Since the cost for computing each local harmonic extension is optimal, the complexity for computing all the harmonic basis functions of $V_h^0(\Gamma_{ij,\ r}^{(k)})$ and $\tilde{V}_h(\Gamma_{ij,\ r}^{(J)})$ is about

$$\sum_{k=0}^{J-1} \mathrm{M}_k \cdot [(\frac{n_{ij}}{\mathrm{M}_k})^{\frac{3}{2}} \cdot m_{k+1}] + \mathrm{M}_J[(\frac{n_{ij}}{\mathrm{M}_k})^{\frac{3}{2}} \cdot \frac{n_{ij}}{\mathrm{M}_k}]. \tag{7.18}$$

This, together with (7.16) and (7.17), leads to

$$N(J) \lesssim \mathrm{N} \times (\frac{H}{h})^3 + \mathrm{N} \times \frac{H}{h} + \mathrm{N} \times \{\sum_{k=0}^{J-1} \mathrm{M}_k \cdot [(\frac{n_{ij}}{\mathrm{M}_k})^{\frac{3}{2}} \cdot m_{k+1}] + \mathrm{M}_J[(\frac{n_{ij}}{\mathrm{M}_k})^{\frac{3}{2}} \cdot \frac{n_{ij}}{\mathrm{M}_k}]\}. \tag{7.19}$$

Using the choices of J and m_k, yields

$$m \cdot n_{ij}^{\frac{3}{2}} = m_1 \cdot n_{ij}^{\frac{3}{2}} \eqsim \mathrm{M}_1 \cdot [(\frac{n_{ij}}{\mathrm{M}_1})^{\frac{3}{2}} \cdot m_2] \eqsim \cdots \eqsim \mathrm{M}_{J-1} \cdot [(\frac{n_{ij}}{\mathrm{M}_{J-1}})^{\frac{3}{2}} \cdot m_J].$$

Moreover, one can verify, by (7.14), that

$$\mathrm{M}_J[(\frac{n_{ij}}{\mathrm{M}_J})^{\frac{3}{2}} \cdot \frac{n_{ij}}{\mathrm{M}_J}] = m \cdot n_{ij}^{\frac{3}{2}}.$$

Thus, we deduce by (7.19)

$$N(J) \lesssim \mathrm{N}[J \cdot m \cdot n_{ij}^{\frac{3}{2}}].$$

Note that $n_{ij} \mathrel{\overset{=}{\sim}} (H/h)^2$, and m is a constant. Using the above inequality, together with (7.14), gives (7.15).

□

Remark 7.2 *Proposition 7.1 indicates that the number J in Theorem 7.1 is much less than* $\log(H/h)$. *For a finite J, one can also choose m_k by*

$$m_1 \cdot n_{ij}^{\frac{3}{2}} \mathrel{\overset{=}{\sim}} \mathrm{M}_1 \cdot [(\frac{n_{ij}}{\mathrm{M}_1})^{\frac{3}{2}} \cdot m_2] \mathrel{\overset{=}{\sim}} \cdots \mathrel{\overset{=}{\sim}} \mathrm{M}_{J-1} \cdot [(\frac{n_{ij}}{\mathrm{M}_{J-1}})^{\frac{3}{2}} \cdot m_J] \mathrel{\overset{=}{\sim}} \mathrm{M}_J[(\frac{n_{ij}}{\mathrm{M}_J})^{\frac{3}{2}} \cdot \frac{n_{ij}}{\mathrm{M}_J}],$$

which makes (7.18) reaches its minimal value. For example, we set $J = 2$. Then, we have

$$m_1 = [n_{ij}^{\frac{4}{19}}], \quad m_2 = [n_{ij}^{\frac{6}{19}}] \quad \textit{and} \quad N(J) \lesssim \mathrm{N} \times (H/h)^{3+\frac{8}{19}},$$

which is much less than the computational complexity in Algorithm 6.1 (compare Remark 6.2). For a given positive $m \geq 2$, one can set $J \mathrel{\overset{=}{\sim}} \log_m(H/h)$, and choose $m_1 = m_2 = \cdots = m_J = m$. In this case, we have $N(J) = O((H/h)^3)$.

8 A two-level substructuring method

In this section, we consider particular local solvers B_k to reduce the use of harmonic basis functions involved in the last section.

8.1 Two-level domain decomposition

We further decompose Ω_k into the union of N_k polyhedrons $\Omega_{k,1}, \cdots, \Omega_{k,\mathrm{N}_k}$, which satisfy $\Omega_{k,i} \cap \Omega_{k,j} = \emptyset$ when $i \neq j$. We assume that each $\partial\Omega_{k,r}$ can be written as a union of boundaries of elements in $\mathcal{T}_h$, and all $\Omega_{k,\ r}$ are of size d in the usual sense. For two neighboring $\Omega_{i,r}$ and $\Omega_{j,l}$, we assume that $\partial\Omega_{i,r} \cap \partial\Omega_{j,l}$ is a common vertex, edge or face of the two subdomains. We define the local finite element spaces $V_h(\Omega_{k,\ r})$ and $V_h^p(\Omega_{k,\ r})$ in the same way with $V_h(\Omega_k)$ and $V_h^p(\Omega_k)$ in Section 6, respectively. For each k, let Γ_k denote the interface associated with the domain decomposition

$$\bar{\Omega}_k = \bigcup_{r=1}^{\mathrm{N}_k} \bar{\Omega}_{k,r} \quad (k = 1, \cdots, \mathrm{N}), \tag{8.1}$$

and let $W_h(\Gamma_k)$ denote the corresponding interface space.

For a face Γ_{ij}, let $\{\Omega_{i,\ i_r}\}$ and $\{\Omega_{j,\ j_r}\}$ $(r = 1, \cdots, \mathrm{M}_{ij})$ denote all the subdomains closing Γ_{ij}. Set

$$\Gamma_{ij,\ r} = \partial\Omega_{i,\ i_r} \cap \partial\Omega_{j,\ j_r} \quad (r = 1, \cdots, \mathrm{M}_{ij}).$$

Assume that $\Gamma_{ij,\ r}$ is just the common face of $\Omega_{i,\ i_r}$ and $\Omega_{j,\ j_r}$. Then, Γ_{ij} can be decomposed into the union of $\Gamma_{ij,\ 1}, \cdots, \Gamma_{ij,\ \mathrm{M}_{ij}}$. Let J_0 be a small positive integer (for example, $j_0 = 2$). As in Subsection 4.1, we make the J_0 level decomposition for Γ_{ij}, such that the J_0 level sub-face $\Gamma_{ij,\ r}^{(J_0)}$ is just $\Gamma_{ij,\ r}$ $(r = 1, \cdots, \mathrm{M}_{J_0} = \mathrm{M}_{ij};\ d_{J_0} = d)$. Namely,

$$\Gamma_{ij} = \bigcup_{r=1}^{\mathrm{M}_1} \Gamma_{ij,\ r}^{(1)} = \bigcup_{r=1}^{\mathrm{M}_2} \Gamma_{ij,\ r}^{(2)} = \cdots = \bigcup_{r=1}^{\mathrm{M}_{J_0}} \Gamma_{ij,\ r}^{(J_0)}.$$

Moreover, we choose auxiliary polyhedrons $\Omega_{ij,\ r}^{(J_0)}$ to be $\Omega_{i,\ i_r} \cup \Omega_{i,\ i_r} \cup \Gamma_{ij,\ r}$ $(r = 1, \cdots, \mathrm{M}_{J_0})$, which satisfy the assumption in Subsection 7.1 (with $d_{J_0} = d$).

8.2 Preconditioner

Let B_0 and $B_{\rm E}$ be defined as in Subsection 6.2. But, define each $B_k = \hat{B}_k$ to be the BPS-type substructuring preconditioner (see B_h in Subsection 6.2) associated with the domain decomposition (8.1). Here, we use the exact local solver $B_{k,\ r}$ on each $V_h^p(\Omega_{k,\ r})$, since each $\Omega_{k,\ r}$ is much smaller than Ω_k. Moreover, the local interface preconditioner on $W_h(\Gamma_k)$ is defined as in Section 5. It follows by Theorem 6.1 that such $\hat{B}_k$ satisfies

$$\int_{\Omega_k} \omega|\nabla v_h|^2 dp \lesssim (\hat{B}_k v_h, v_h)_{\Omega_k} \lesssim [1+\log(d/h)]^2 \int_{\Omega_k} \omega|\nabla v_h|^2 dp, \quad \forall v_h \in V_h^p(\Omega_k), \tag{8.2}$$

instead of (6.5).

Let $B^{(l)}_{ij,\ r,\ 0}$ $(l = 0, \cdots, J_0 - 1)$ and $\tilde{B}^{(J_0)}_{ij,\ r}$ be defined as in Subsection 7.2.

Now, we define the two-level substructuring preconditioner for A_h (but with multilevel interface solvers) by

$$\begin{aligned} \hat{B}_{J_0}^{-1} &= \sum_{k=1}^{\rm N} \hat{B}_k^{-1} Q_k^* + B_0^{-1} Q_0^* + \sum_{{\rm E}\in\mathcal{E}_\Gamma} B_{\rm E}^{-1} Q_{\rm E}^* \\ &+ \sum_{\Gamma_{ij}} [\sum_{k=0}^{J_0-1} \sum_{r=1}^{{\rm M}_k} (B^{(k)}_{ij,\ r,\ 0})^{-1} Q^{(k),*}_{ij,\ r,\ 0} + \sum_{r=1}^{{\rm M}_{J_0}} (\tilde{B}^{(J_0)}_{ij,\ r})^{-1} Q^{(J),*}_{ij,\ r}]. \end{aligned}$$

In an analogous way with Theorem 7.1, one can prove (using (8.2))

Theorem 8.1 *For the preconditioner $\hat{B}_{J_0}$, we have*

$$cond(\hat{B}_{J_0}^{-1} A_h) \le C J_0^3 [1+\log(d/h)]^2 [1+\log(H/h)]^2, \tag{8.3}$$

where C is a constant independent of h, H, d_k and the jumps of the coefficient ω across the faces Γ_{ij}. Here, the level $J_0 \ll \log_{\frac{3}{2}}(\log(H/h))$, and the refined size $d \ll H$.

□

8.3 Implementation

Since $B_{k,\ r}$ in $\hat{B}_k$ is exact, the action of $\hat{B}_{J_0}^{-1}$ can be described by the following algorithm (refer to Algorithm 6.2 and Algorithm 7.1)

Algorithm 8.1. For a $g \in V_h(\Omega)$, the solution $u_g \in V_h(\Omega)$ satisfying

$$(\hat{B}_{J_0} u_g, v_h) = (g, v_h), \quad \forall v_h \in V_h(\Omega)$$

can be gotten as follows:

Step 1′. Computing $u_k^p \in V_k^p(\Omega_k)$ in parallel by Algorithm 6.2 associated with the domain decomposition (8.1);

Step 2. Computing $\phi_0 \in W_h^0(\Gamma)$ by

$$\langle M_0 \phi_0, \psi_h \rangle_\Gamma = (g, v_h), \quad v_h = E_0 \psi_h, \quad \forall \psi_h \in W_h^0(\Gamma);$$

Step 3. Computing $\phi_{\rm E} \in \tilde{W}_h({\rm E})$ in parallel by

$$\langle M_{\rm E} \phi_{\rm E}, \psi_h \rangle_{\rm E} = (g, v_h)_{\Omega_{\rm E}}, \quad v_h = E_{\rm E} \psi_h, \quad \forall \psi_h \in \tilde{W}_h({\rm E});$$

Step 4. Computing $\phi^{(k)}_{ij,\ r,\ 0} \in W_h^0(\Gamma^{(k)}_{ij,\ r})$ in parallel by

$$\langle M^{(k)}_{ij,\ r,\ 0} \phi^{(k)}_{ij,\ r,\ 0}, \psi_h \rangle_{\Gamma^{(k)}_{ij,\ r}} = (g, v_h)_{\Omega^{(k)}_{ij,\ r}}, \quad v_h = E^{(k)}_{ij,\ r,\ 0} \psi_h, \quad \forall \psi_h \in W_h^0(\Gamma^{(k)}_{ij,\ r});$$

Step 5′. Computing $\phi^{(J_0)}_{ij,\ r}\in \tilde{W}_h(\bar{\Gamma}^{(J_0)}_{ij,\ r})=\tilde{W}_h(\bar{\Gamma}_{ij,\ r})$ in parallel by

$$\langle M^{(J_0)}_{ij,\ r}\phi^{(J_0)}_{ij,\ r},\psi_h\rangle_{\Gamma^{(J_0)}_{ij,\ r}} = (g,\tilde{\psi}_h)_{\Omega_{ij}}-\sum_{k=i,j}\omega_k(\nabla u^p_k,\nabla\tilde{\psi}^*_h)_{\Omega_k},$$
$$\psi^*_h=I^0_{\Gamma_{ij,\ r}}\psi_h,\ \forall\psi_h\in\tilde{W}_h(\bar{\Gamma}_{ij,\ r}).$$

Here, we assume that $\Gamma^{(J_0)}_{ij,\ r}$ is just $\Gamma_{ij,\ r}$;

Step 6. Computing

$$v_0=E_0\phi_0,\ \ v_{\rm E}=E_{\rm E}\phi_{\rm E},\ \ v^{(l)}_{ij,\ r,\ 0}=E^{(l)}_{ij,\ r,\ 0}\phi^{(l)}_{ij,\ r,\ 0}\ \ (l=0,\cdots,J-1)$$

and

$$\tilde{v}^{(J_0)}_{ij,\ r}=E^{(J)}_{ij,\ r}\phi^{(J_0)}_{ij,\ r}.$$

Step 7. Set

$$u_g = \sum_{k=1}^{\rm N}v^p_k+v_0+\sum_{{\rm E}\in\mathcal{E}_\Gamma}v_{\rm E}+\sum_{\Gamma_{ij}}[\sum_{l=1}^{J_0-1}\sum_{r=1}^{{\rm M}_k}v^{(l)}_{ij,\ r,\ 0}+\sum_{r=1}^{{\rm M}_{J_0}}\tilde{v}^{(J)}_{ij,\ r}].$$

Remark 8.1 *In Algorithm 8.1, both coarser spaces and the local spaces $V^p_h(\Omega_{k,r})$ has low dimensions (compare Section 6), and so we can use exact local solvers $B_{k,\ r}$ on each $\Omega_{k,\ r}$. Moreover, we need not to computing the harmonic basis of $V_h(\Omega^{(J_0)}_{ij,\ r})$ before implementing Step 5′ above (compare Step 5 in Algorithm 7.1). Because of this, it is allowed that the local subspace $\tilde{W}_h(\bar{\Gamma}^{(J_0)}_{ij,\ r})$ possesses slightly high dimensions. This means that the level J_0 may be a small positive integer.*

9 Approximate harmonic extensions

In this section, we discuss how to get the basis of the spaces $V^0_h(\Omega)$, $V_h(\Omega_{\rm E})$, $V^0_h(\Omega^{(k)}_{ij,\ r}$ $(k=0,\cdots,J-1)$ and $\tilde{V}_h(\Omega^{(J)}_{ij,\ r}$. One needs to define suitable harmonic extensions E_0, $E_{\rm E}$, $E^{(k)}_{ij,\ r,\ 0}$ $(k=0,\cdots,J-1)$ and $\tilde{E}^{(J)}_{ij,\ r}$.

It is known that $E_{\rm E}$ can be defined to be the zero extension. Namely,

$$(E_{\rm E}\phi_h)(p)=\begin{cases}\phi_h(p), & \text{if } p\in\mathcal{N}_h\cap{\rm E},\\ 0, & \text{if } p\in\mathcal{N}_h\cap(\Omega\backslash{\rm E}).\end{cases}$$

Then, $\{E_{\rm E}\varphi^l_{\rm E}\}$ constitute a basis of $V_h(\Omega_{\rm E})$, where $\{\varphi^l_{\rm E}\}$ is the nodal basis of $\tilde{W}_h({\rm E})$. Such extension satisfies

$$\sum_{k\in\mathcal{Q}_{\rm E}}\omega_k|E_{\rm E}\phi_h|^2_{1,\Omega_k}\ \overline{\gtrsim}\ \|\phi_h\|^2_{*,\Gamma},\quad \forall\phi_h\in\tilde{W}_h({\rm E}).$$

To define the other extensions, we have to develop particular techniques.

For convenience, set

$$W^0_h(\partial\Omega_k)=\{\varphi_h|_{\partial\Omega_k}:\ \varphi_h\in W^0_h(\Gamma)\}\ \ (k=1,\cdots,{\rm N}).$$

Assume that there is a suitable extension $E_k:\ W^0_h(\partial\Omega_k)\rightarrow V_h(\Omega_k)$, such that

$$|E_k\varphi_h|_{1,\ \Omega_k}\lesssim|\varphi_h|_{\frac{1}{2},\ \partial\Gamma_k},\ \ \forall\varphi_h\in W^0_h(\partial\Gamma_k).$$

Then, we define $E_0: \ W_h^0(\Gamma) \to V_h(\Omega)$ by

$$(E_0\varphi_h)|_{\Omega_k} = E_k(\varphi_h|_{\partial\Omega_k}), \quad \forall \varphi_h \in W_h^0(\Gamma) \quad (k = 1, \cdots, \mathrm{N}).$$

For a $\Omega^{(k)}_{ij,\ r}$, set

$$\Omega^{(k)}_{ij,\ r_i} = \Omega^{(k)}_{ij,\ r} \cap \Omega_i.$$

It is known that

$$|\varphi_h|_{H_{00}^{\frac{1}{2}}(\Gamma^{(k)}_{ij,\ r})} \eqsim |I^0_{\bar{\Gamma}^{(k)}_{ij,\ r}}\varphi_h|_{\frac{1}{2},\ \partial\Omega^{(k)}_{ij,\ r_i}}, \quad \varphi_h \in W_h^0(\Gamma^{(k)}_{ij,\ r}) \ (k = 0, \cdots, J-1), \text{ or } \varphi_h \in \tilde{W}_h(\Gamma^{(J)}_{ij,\ r}).$$

Then, the extensions $E^{(k)}_{ij,\ r,\ 0}$ and $E^{(J)}_{ij,\ r}$ should satisfy

$$|E^{(k)}_{ij,\ r,\ 0}\varphi_h|_{1,\ \Omega^{(k)}_{ij,\ r_i}} \lesssim |I^0_{\bar{\Gamma}^{(k)}_{ij,\ r}}\varphi_h|_{\frac{1}{2},\ \partial\Omega^{(k)}_{ij,\ r_i}}, \quad \forall \varphi_h \in W_h^0(\Gamma^{(k)}_{ij,\ r}) \quad (k = 0, \cdots, J-1),$$

and

$$|E^{(J)}_{ij,\ r}\varphi_h|_{1,\ \Omega^{(J)}_{ij,\ r_i}} \lesssim |I^0_{\bar{\Gamma}^{(J)}_{ij,\ r}}\varphi_h|_{\frac{1}{2},\ \partial\Omega^{(J)}_{ij,\ r_i}}, \quad \forall \varphi_h \in \tilde{W}_h(\Gamma^{(J)}_{ij,\ r}).$$

For ease of notation, let $\hat{\Omega}$ denote Ω_k or the representative of all $\Omega^{(k)}_{ij,\ r_i}$ $(k = 0, \cdots, J;\ r = 1, \cdots, \mathrm{M}_J)$. It suffices to define a suitable extension $\hat{E}_h: \ W_h(\partial\hat{\Omega}) \to V_h(\hat{\Omega})$, which satisfies

$$|\hat{E}_h\varphi_h|_{1,\ \hat{\Omega}} \lesssim |\varphi_h|_{\frac{1}{2},\ \partial\hat{\Omega}}, \quad \forall \varphi_h \in W_h(\hat{\Gamma}). \tag{9.1}$$

The desired extensions will be defined by the harmonic extension onto a ball.

9.1 Mapping from the reference domain onto ball

Let $\hat{\Omega}$ be a reference polyhedron with size $\hat{d}$, and let $\hat{\mathcal{T}}_h$ denote the triangulation on $\hat{\Omega}$. Consider the ball

$$U = \{(x, y, z): \ x^2 + y^2 + z^2 \leq R^2\},$$

where R is a sufficiently small positive number, such that the ball U contains $\hat{\Omega}$ as its interior. It is clear that R almost equals to $\hat{d}$.

We need to define a mapping F, which maps $\hat{\Omega}$ onto U.

At first, we choose an element $\hat{\triangle}$ of $\hat{\mathcal{T}}_h$, such that this element contains the barycenter of $\hat{\Omega}$. Then, we move $\hat{\Omega}$, such that the barycenter of the element $\hat{\triangle}$ coincides the origin point o.

For each node p of $\hat{\mathcal{T}}_h$, draw a line l from o to p (note that $p \neq o$). Let $\hat{p}$ denote the intersection of l with $\partial\hat{\Omega}$, and define (refer to [23])

$$F(p) = \frac{R}{|o\hat{p}|}p, \quad \forall p \in \mathcal{N}_h \cap \hat{\Omega}.$$

It is easy to see that $F(\hat{p})$ is just the intersection of l with ∂U. Namely, F maps each node on $\partial\hat{\Omega}$ to a point on ∂U. It is easy to see that all the points $\{F(p)\}$ $(p \in \mathcal{N}_h \cap \partial\hat{\Omega})$ can generate a (curve) triangulation on ∂U. We require that this triangulation, which is denoted by $\mathcal{T}_h^{\partial U}$, becomes the standard (straight line) triangulation on $[0, 2\pi] \times [0, \pi]$ under the ball coordinate system.

To give a clearer expression of $F(p)$, we decompose $\hat{\Omega}$ into union of several tetrahedrons $\hat{\Omega}_1, \cdots, \hat{\Omega}_m$, such that the origin point o is the common vertex of all $\hat{\Omega}_r$ $(r = 1, \cdots, m)$. Let $\{A_i^r\}_{i=1}^3$ be the other vertices of $\hat{\Omega}_r$. We assume that each A_i^r is just a vertex of $\hat{\Omega}$, moreover, all $\{A_i^r\}_{i=1}^3$ (with fixed r) are on a same face of $\partial\hat{\Omega}$. For a node $p \in \hat{\Omega}_r$, let λ_0

and λ_i^r denote the barycenter coordinates of p at the origin point o and A_i^r $(r = 1, 2, 3)$, respectively. Then, we have (refer to [10])

$$F(p) = \frac{R\lambda_0 \sum_{i=1}^{3} \lambda_i^r A_i^r}{|\sum_{i=1}^{3} \lambda_i^r A_i^r|}, \quad \forall p \in \mathcal{N}_h \cap \hat{\Omega}_r \;\; (r = 1, \cdots, m).$$

9.2 Extension into ball

With the triangulation $\mathcal{T}_h^{\partial U}$, let $W_h(\partial U)$ denote the linear or bilinear finite element space on ∂U. For $\phi_h \in W_h(\partial U)$, let $E_U\phi_h \in H^1(U)$ be the solution of the Laplace equation:

$$\begin{cases} \triangle u^* = 0, & \text{in } U, \\ u^* = \phi_h, & \text{on } \partial U. \end{cases}$$

Using a similar technique with [6], one can prove that the extension $E_U\phi_h$ can be written as (refer to [37])

$$(E_U\phi_h)(r,\theta,\varphi) = \sum_{k=0}^{+\infty} \frac{2k+1}{4\pi} (\frac{r}{R})^k \int_0^{2\pi} \int_0^{\pi} L_k(cos\ \alpha)\phi_h(\sigma,\psi) sin\ \sigma d\sigma d\psi, \quad (r,\theta,\varphi) \in U, \tag{9.2}$$

where L_k is the k-th Legendre polynomial, and

$$cos\ \alpha = cos\ \theta \cdot cos\ \sigma + sin\ \theta \cdot sin\ \sigma \cdot cos(\varphi - \psi).$$

It is well known that

$$|E_U\phi_h|_{1,U} \lesssim |\phi_h|_{\frac{1}{2},\partial U}, \quad \forall \phi_h \in W_h(\partial U), \tag{9.3}$$

and

$$\|E_U\phi_h\|_{1+s,U} \lesssim \|\phi_h\|_{\frac{1}{2}+s,\partial U}, \quad \forall \phi_h \in W_h(\partial U) \;\; (0 \leq s < 1). \tag{9.4}$$

9.3 The desired extensions

With the triangulation $\hat{\mathcal{T}}_h$, let $V_h(\hat{\Omega})$ and $W_h(\partial\hat{\Omega})$ denote the linear finite element space on $\hat{\Omega}$ and on $\partial\hat{\Omega}$, respectively. In this subsection, we will apply the mapping F and the extension E_U to defining an approximate harmonic extension $\hat{E} : W_h(\partial\hat{\Omega}) \rightarrow V_h(\hat{\Omega})$.

It suffices to define the value of $(\hat{E}\phi_h)(p)$ for each $p \in \mathcal{N}_h \cap \hat{\Omega}$.

For a $p \in \mathcal{N}_h \cap \partial\hat{\Omega}$, define $F\phi_h \in W_h(\partial U)$ by

$$(F\phi_h)(F(p)) = \phi_h(p).$$

For $p \in \mathcal{N}_h \cap \hat{\Omega}$, let $(r_p, \theta_p, \varphi_p)$ denote the ball coordinate of $F(p)$. Define

$$(\hat{E}\phi_h)(p) = (E_U F\phi_h)(F(p)).$$

Namely,

$$(\hat{E}\phi_h)(p) = \sum_{k=0}^{+\infty} \frac{2k+1}{4\pi} (\frac{r_p}{R})^k \int_0^{2\pi} \int_0^{\pi} L_k(cos\ \alpha_p)(F\phi_h)(\sigma,\psi) sin\ \sigma d\sigma d\psi, \quad \forall p \in \mathcal{N}_h \cap \hat{\Omega}, \tag{9.5}$$

where

$$cos\ \alpha_p = cos\ \theta_p \cdot cos\ \sigma + sin\ \theta_p \cdot sin\ \sigma \cdot cos(\varphi_p - \psi).$$

Theorem 9.1 *Let $\hat{E}_h$ be the extension defined by (9.5). Then,*

$$|\hat{E}_h\phi_h|_{1,\hat{\Omega}} \lesssim |\phi_h|_{\frac{1}{2},\partial\hat{\Omega}}, \quad \forall\phi_h \in W_h(\partial\hat{\Omega}). \tag{9.6}$$

Proof. It is easy to see that all the nodes $\{F(p)\}$ with $p \in \mathcal{N}_h \cap (\hat{\Omega} \cup \partial\hat{\Omega})$ constitute a triangulation on U. Let $V_h(U)$ denote the "linear" finite element space associated with this induced triangulation. Here, the basis functions on the curve elements closing ∂U are defined in the standard manner. As usual, we define the interpolation operator $\pi_h : C(U) \to V_h(U)$. By the definition of $\hat{E}_h$, and using the discrete norm, yields

$$|\hat{E}_h\phi_h|_{1,\hat{\Omega}} \eqsim |\pi_h(E_U F\phi_h)|_{1,U}. \tag{9.7}$$

Since $F\phi_h \in H^{\frac{1}{2}+s}(\partial U)$ for $s \in [0,1)$, we have $E_U F\phi_h \in H^{1+s}(U)$. Thus, we have for $s \in (\frac{1}{2}, 1)$

$$|\pi_h(E_U F\phi_h)|_{1,U} \leq |E_U F\phi_h|_{1,U} + |(\pi_h - I)E_U F\phi_h|_{1,U} \lesssim |E_U F\phi_h|_{1,U} + h^s|E_U F\phi_h|_{1+s,U}.$$

This, together with (9.3)-(9.4), leads to

$$|\pi_h(E_U F\phi_h)|_{1,U} \lesssim |F\phi_h|_{\frac{1}{2},\partial U} + h^s|F\phi_h|_{\frac{1}{2}+s,\partial U} \lesssim |F\phi_h|_{\frac{1}{2},\partial U}.$$

Here, we have used the inverse estimate. Substituting the above inequality into (9.7), and using the discrete norm again, we deduce (9.6).
□

Remark 9.1 *The main task for getting the extension $\hat{E}_h$ is to compute approximately the integrations in (9.5). Since only the stability of $\hat{E}_h$ is required, such integrations can be calculated roughly. Moreover, one needs only to compute a few terms of the series (9.5). One can also replace (9.5) by a singular integration given in [37]. The method introduced in section is a general method, which can be extended to Maxwell's equations. The mapping F was used widely in finite element methods on non-polyhedron domain, see, for example, [10]. A similar mapping has been used to construct an approximate harmonic extension on polygonal domain in [23].*

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