A Backward Euler-DtN Alternating Method for Time-dependent Eddy Current Problems in Unbounded Domains

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Abstract

In this paper, we develop a coupling of natural boundary element method and finite element method for solving the time-dependent eddy current problems in unbounded domains based on a $H-\psi$ formulation. The $H-\psi$ formulation allows that a scalar function is used outside the conductor and a vector function is only used in the conducting domain. To get a full discrete scheme of the problem, the backward Euler method is applied for the discretization of the time variable and the Nédélec element of the lowest order, the piecewise linear element and the curvilinear element are used for the discretization of the space variables. A backward Euler-DtN alternating method is designed to solve the discrete coupled problem. In each time step of the method, the action of the boundary operators can be implemented by calculating a series of the spherical harmonics, instead of solving boundary integral equations. We derive an error estimate of the fully discrete scheme, and prove the convergence of the alternating method. Numerical experiments show the effectiveness of the new method.

Key words. eddy current problems, unbounded domains, boundary integrations, Nédélec finite elements, non-overlapping domain decomposition.

AMS(MOS) subject classification. 65N30, 65N55

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1 Introduction

The eddy currents problems, which are described by Maxwell's equations without the displacement currents [1, 5, 6], are popular in electromagnetism. There are many researches on solutions of eddy current problems in literature. For example, the boundary integral equation methods [15, 16, 22], the mixed finite element method [2] and the mixed finite element method associated with boundary element [7]. In addition, there are some other particular methods, such as mixed FEM-BIEM method [8], the adaptive finite element method [32], symmetric FEM-BEM method [17] and mortar element method [14, 23].

In this paper, based on the ideas described in [30, 31] and the eddy current formulation of the magnetic field ($H-\psi$ formulation), we develop a coupling of natural boundary element method and finite element method for solving the time-dependent eddy current problems in unbounded domains. The $H-\psi$ formulation allows that a scalar function is used outside the conductor and a vector function is only used in the conducting domain. We first introduce a spherical artificial boundary which contains the conductor, so the whole unbounded domain is decomposed into three subdomains by the surface of the conductor and the spherical artificial boundary. Then the backward Euler method is applied for the discretization of time variable, and the coupled variational problem at each time step can be derived by using the natural integral operator of the harmonic equation for the exterior spherical domain. For the discretization of the space variables, the Nédélec element of the lowest order is introduced in the conductor, while the piecewise linear element and the curvilinear element closing the spherical boundary are used in the domain which is between the spherical artificial boundary and the surface of the conductor. A backward Euler-DtN alternating algorithm is designed to solve the corresponding discrete problem. In this new method, the action of the natural integral operator can be implemented by calculating a series of the spherical harmonics, so there is no need to solve boundary integral equations. We derive an error estimate for the fully discrete scheme, and prove the convergence of the alternating algorithm. Some numerical experiments will illustrate our theoretical results.

The paper is arranged as follows. The eddy current problems are described in Section 2. The coupled variational problem at each time step is given in Section 3. Section 4 presents the discretizations and introduces a backward Euler-DtN alternating algorithm. In Section 5, we derive an error estimate for the fully discrete scheme, and prove the convergence of the backward Euler-DtN alternating algorithm. Section 6 discusses some implementation details of the algorithm. Numerical examples are given in section 7. In Section 8 we summarize the backward Euler-DtN alternating algorithm.

2 Preliminaries

2.1 The model

Let $\Omega_c \subset \mathbb{R}^3$ be a conducting domain and Γ_c be its boundary. The external domain is denoted by $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}_c$. The eddy currents problems in $\mathbb{R}^3$ can be described as

$$\begin{cases} \nabla \times \mathbf{H} = \mathbf{J}, & \text{in } \mathbb{R}^3, \\ \mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0, & \text{in } \mathbb{R}^3, \\ \nabla \cdot \mathbf{B} = 0, & \text{in } \mathbb{R}^3, \end{cases} \tag{2.1}$$

where $\mathbf{H}$, $\mathbf{B}$, $\mathbf{J}$ and $\mathbf{E}$ denote the magnetic field, the magnetic flux density, the total current density and the electric field, respectively. $\mathbf{B} = \mu\mathbf{H}$ and μ is the magnetic permeability. $\mathbf{J}$ can be defined by

$$\mathbf{J} = \begin{cases} \sigma\mathbf{E}, & \text{in } \Omega_c, \\ \mathbf{J}_s, & \text{in } \Omega_e, \end{cases} \tag{2.2}$$

where $\mathbf{J}_s$ is the solenoidal source current carried by some coils in the air and σ stands for the electric conductivity. We assume that $\mu, \sigma \in L^\infty(\mathbb{R}^3)$, which are time independent and associated with linear isotropic media, and there exist two positive constants μ_{min} and σ_{min} such that $\mu \geq \mu_{min}$ in $\mathbb{R}^3$, $\sigma \geq \sigma_{min}$ in Ω_c and $\sigma \equiv 0$ in Ω_e.

The problem has the following interface conditions:

$$[\mathbf{H}] \times \mathbf{n}_c = 0, \qquad [\mathbf{B}] \cdot \mathbf{n}_c = 0, \qquad \text{on } \Gamma_c, \tag{2.3}$$

where $[\mathbf{v}]$ stands for the jump of $\mathbf{v}$ at the interface and $\mathbf{n}_c$ denotes the unit outward normal vector on Γ_c. Moreover, such a type of interface conditions is to be verified at any surface where σ or μ is discontinuous [23]. Besides the interface conditions, we have to impose the appropriate regularity conditions at infinity. Since the problem is time-dependent, the suitable initial conditions are also needed. In particular, the initial condition on $\mathbf{B}$ has to satisfy $\nabla \cdot \mathbf{B} = 0$ and $[\mathbf{B}] \cdot \mathbf{n} = 0$ at any interface. It is easy to see that the condition $\nabla \cdot \mathbf{B} = 0$ is satisfied at any time, provided that this condition is met at the initial time [14].

2.2 The Sobolev spaces

In order to describe the problem in a mathematically rigorous way, this subsection is devoted to the definitions of function spaces.

Function spaces in Ω_c and Ω_e. Assume that Ω_c is a bounded simply connected convex polyhedron domain. We recall the Sobolev spaces $H^s(\Omega_c)$, $s \geq 0$ with the convention $H^0 \equiv L^2$. $L^2(\Omega_c)$ is the usual Hilbert space of square integrable functions with the norm $\|u\|_{L^2(\Omega_c)} = \left(\int_{\Omega_c} u^2 dx\right)^{\frac{1}{2}}$.

$$H^s(\Omega_c) := \{u \in L^2(\Omega_c) \mid D^\xi u \in L^2(\Omega_c), |\xi| \leq s\}$$

endowed with the norm and semi-norm

$$\|u\|_{H^s(\Omega_c)} := \left(\sum_{|\xi|\leq s} \|D^\xi u\|^2_{L^2(\Omega_c)}\right)^{\frac{1}{2}} \quad \text{and} \quad |u|_{H^s(\Omega_c)} := \left(\sum_{|\xi|=s} \|D^\xi u\|^2_{L^2(\Omega_c)}\right)^{\frac{1}{2}},$$

where ξ represents non-negative triple index. From now on, we denote vector-valued quantities by boldface notation, such as $\mathbf{L}^2(\Omega_c) := (L^2(\Omega_c))^3$. Define

$$\mathbf{H}(\mathbf{curl}, \Omega_c) := \{\mathbf{u} \in \mathbf{L}^2(\Omega_c) \mid \mathbf{curl}\mathbf{u} \in \mathbf{L}^2(\Omega_c)\},$$

$$\mathbf{H}_0(\mathbf{curl}, \Omega_c) := \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega_c) \mid \mathbf{u} \times \mathbf{n}_c = 0 \text{ on } \Gamma_c\}.$$

$\mathbf{H}(\mathbf{curl}, \Omega_c)$ is equipped with the norm:

$$\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega_c)} := \left(\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega_c)} + \|\mathbf{curl}\mathbf{u}\|^2_{\mathbf{L}^2(\Omega_c)}\right)^{\frac{1}{2}}.$$

Moreover, we define

$$\mathbf{W}^1_0(\Omega_e) := \left\{\varphi \mid \frac{\varphi}{r} \in L^2(\Omega_e), \nabla\varphi \in \mathbf{L}^2(\Omega_e)\right\},$$

equipped with the norm

$$\|\varphi\|_{\mathbf{W}^1_0(\Omega_e)} := \left(\|\frac{\varphi}{r}\|^2_{L^2(\Omega_e)} + \|\nabla\varphi\|^2_{\mathbf{L}^2(\Omega_e)}\right)^{\frac{1}{2}},$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

Function spaces on the boundary. In the paper, we also need some Sobolev spaces defined on the conductor surface Γ_c and a spherical artificial boundary Γ_e. Let Γ denote Γ_e or Γ_c, then the Sobolev spaces $H^s(\Gamma)$, $H^s_t(\Gamma)$ and differential surface operators ($curl_\Gamma$, $\mathbf{curl}_\Gamma$, div_Γ and ∇_Γ) can be defined for all $s \in \mathbb{R}$ using local charts and transformations [9, 11].

For any $s > 0$, Let

$$\mathbf{H}^s_t(\Gamma_c) = \{\mathbf{v} \in \mathbf{H}^s(\Gamma_c) \quad | \ \mathbf{v} \cdot \mathbf{n}_c = 0 \quad \text{on} \quad \Gamma_c\} \subset \mathbf{L}^2_\mathrm{t}(\Gamma_\mathrm{c}) \equiv \mathbf{H}^0_\mathrm{t}(\Gamma_\mathrm{c}),$$

whose norm is defined in [24]. $\mathbf{H}^{-s}_t(\Gamma_c)$ denotes the dual space of $\mathbf{H}^s_t(\Gamma_c)$ with $\mathbf{L}^2_t(\Gamma_c)$ as a pivot space. The norm on $\mathbf{H}^{-s}_t(\Gamma_c)$ can be written as

$$\|\mathbf{v}\|_{\mathbf{H}^{-s}_t(\Gamma_c)} = \sup_{\mathbf{u}\in\mathbf{H}^s_t(\Gamma_c)} \frac{|\int_{\Gamma_c} \mathbf{v} \cdot \overline{\mathbf{u}} dA|}{\|\mathbf{u}\|_{\mathbf{H}^s_t(\Gamma_c)}}.$$

Moreover, we know

$$div_{\Gamma_c}((\mathbf{u} \times \mathbf{n}_c)|_{\Gamma_c}) = (\mathbf{curl}\mathbf{u}) \cdot \mathbf{n}_c, \quad \forall\ \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega_c),$$

which implies $div_{\Gamma_c}(\gamma_\tau(\mathbf{u})) \in H^{-\frac{1}{2}}(\Gamma_c)$ [10]. Then we define the space:

$$\mathbf{H}^{-\frac{1}{2}}(div_{\Gamma_c}, \Gamma_c) = \{\mathbf{v} \mid \mathbf{v} \in \mathbf{H}^{-\frac{1}{2}}_t(\Gamma_c), \quad div_{\Gamma_c}\mathbf{v} \in H^{-\frac{1}{2}}(\Gamma_c)\},$$

whose norm is

$$\|\mathbf{v}\|_{\mathbf{H}^{-\frac{1}{2}}(div_{\Gamma_c},\Gamma_c)} = \left(\|\mathbf{v}\|^2_{\mathbf{H}^{-\frac{1}{2}}_t(\Gamma_c)} + \|div_{\Gamma_c}\mathbf{v}\|^2_{H^{-\frac{1}{2}}(\Gamma_c)}\right)^{\frac{1}{2}}.$$

Function spaces involving time. According to [27], we give the definitions of the function spaces involving time. Let $\mathbf{X}$ be a real Banach space with norm $\|\cdot\|_{\mathbf{X}}$. $\mathbf{L}^p((0,T);\mathbf{X})$ consists of all measurable functions $\zeta : [0,T] \to \mathbf{X}$ with

$$\|\zeta\|_{\mathbf{L}^p((0,T);\mathbf{X})} := \left(\int_0^T \|\zeta\|_{\mathbf{X}}^p dt \right)^{\frac{1}{p}} < \infty, \quad \text{for } 1 \le p < \infty,$$

and

$$\|\zeta\|_{\mathbf{L}^\infty((0,T);\mathbf{X})} := \text{ess} \sup_{0 \le t \le T} \|\zeta\|_{\mathbf{X}} < \infty.$$

2.3 The $\mathbf{H} - \psi$ formulation

Since $\mathrm{div}\mathbf{J}_s \equiv 0$, there exists a source magnetic field $\mathbf{H}_s$ such that

$$\mathbf{J}_s = \mathbf{curl}\,\mathbf{H}_s, \quad \text{in } \mathbb{R}^3. \tag{2.4}$$

By using the Biot-Savart Law for general coils, the field $\mathbf{H}_s$ can be written as:

$$\mathbf{H}_s := \mathbf{curl}\,\mathbf{A}_s \quad \text{where} \quad \mathbf{A}_s := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{J}_s(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} dy.$$

From now on, our goal is to find the residual $\mathbf{H}_0 := \mathbf{H} - \mathbf{H}_s$, which is called the reaction field [13, 32].

Since

$$\nabla \times \mathbf{H} = \nabla \times (\mathbf{H}_0 + \mathbf{H}_s) = \mathbf{J}_s, \quad \text{in } \Omega_e,$$

by using (2.4), we have

$$\nabla \times \mathbf{H}_0 = 0, \quad \text{in } \Omega_e, \tag{2.5}$$

which suggests that there exists a scalar potential ψ such that

$$\mathbf{H}_0 = \nabla \psi, \quad \text{in } \Omega_e. \tag{2.6}$$

To simplify the problem, we assume that μ and σ are constant functions. It is known that the problem in Ω_c can be written as:

$$\begin{cases} \nabla \times \mathbf{H} = \sigma \mathbf{E}, & \text{in } \Omega_c, \\ \mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0, & \text{in } \Omega_c, \\ \nabla \cdot \mathbf{H} = 0, & \text{in } \Omega_c. \end{cases} \tag{2.7}$$

By using $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_s$ in Ω_c, we can obtain the following problem with the variables $\mathbf{H}_0$ and $\mathbf{E}$,

$$\begin{cases} \nabla \times \mathbf{H}_0 = \sigma \mathbf{E}, & \text{in } \Omega_c, \\ \mu \frac{\partial \mathbf{H}_0}{\partial t} + \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}_s}{\partial t}, & \text{in } \Omega_c, \\ \nabla \cdot \mathbf{H}_0 = 0, & \text{in } \Omega_c. \end{cases} \tag{2.8}$$

According to the first two equations of (2.8), we can get

$$\mu \frac{\partial \mathbf{H}_0}{\partial t} + \frac{1}{\sigma} \nabla \times \nabla \times \mathbf{H}_0 = -\mu \frac{\partial \mathbf{H}_s}{\partial t} \quad \text{in} \quad \Omega_c. \tag{2.9}$$

Furthermore, the problem has the following interface conditions on Γ_c:

$$\mathbf{H}_0 \times \mathbf{n}_c = \nabla\psi \times \mathbf{n}_c, \qquad \mathbf{H}_0 \cdot \mathbf{n}_c = \nabla\psi \cdot \mathbf{n}_c, \qquad \text{on } \Gamma_\mathrm{c}. \tag{2.10}$$

Since $\nabla \cdot \mathbf{H} = 0$ in $\mathbb{R}^3$, we can get $\triangle\psi = 0$ in Ω_e. Then the problem with the variable ψ in Ω_e is as follows:

$$\begin{cases} \triangle\psi = 0, & \text{in } \Omega_e, \\ \frac{\partial\psi}{\partial\mathbf{n}_c} = \mathbf{H}_0 \cdot \mathbf{n}_c, & \text{on } \Gamma_c. \end{cases} \tag{2.11}$$

According to [33], we know that the problem (2.11) has a unique solution $\psi \in \mathbf{W}_0^1(\Omega_e)$, provided $\mathbf{H}_0 \cdot \mathbf{n}_c$ lies in the space $H^{-\frac{1}{2}}(\Gamma_c)$.

For the initial condition, we set

$$\mathbf{H}_0(\cdot, 0) = \mathbf{0}, \quad \psi(\cdot, 0) = 0.$$

So we can write the initial problem with the variables $\mathbf{H}_0$ and ψ as:

$$\begin{cases} \mu\frac{\partial\mathbf{H}_0}{\partial t} + \frac{1}{\sigma}\nabla\times\nabla\times\mathbf{H}_0 = -\mu\frac{\partial\mathbf{H}_s}{\partial t}, & \text{in } \Omega_\mathrm{c}, \\ \nabla\cdot\mathbf{H}_0 = 0, & \text{in } \Omega_c, \\ \triangle\psi = 0, & \text{in } \Omega_\mathrm{e}, \\ \mathbf{H}_0 \times \mathbf{n}_c = \nabla\psi \times \mathbf{n}_c, & \text{on } \Gamma_\mathrm{c}, \\ \mathbf{H}_0 \cdot \mathbf{n}_c = \nabla\psi \cdot \mathbf{n}_c, & \text{on } \Gamma_\mathrm{c}, \\ \mathbf{H}_0(\mathbf{x}, 0) = 0, & \text{for } \mathbf{x} \in \Omega_\mathrm{c}, \\ \psi(\mathbf{x}, 0) = 0, & \text{for } \mathbf{x} \in \Omega_\mathrm{e}. \end{cases} \tag{2.12}$$

In an analogous way to [32], we can define

$$\begin{aligned} \mathbf{V} := \quad & \{\mathbf{v} \mid \mathbf{v} = \mathbf{w} \text{ in } \Omega_\mathrm{c} \text{ for some } \mathbf{w} \in \mathbf{H}(\mathbf{curl}, \Omega_\mathrm{c}) \text{ and } \mathbf{v} = \nabla\varphi \text{ in } \Omega_\mathrm{e} \\ & \text{for some } \varphi \in \mathbf{W}_0^1(\Omega_\mathrm{e}) \text{ such that } \mathbf{w} \times \mathbf{n}_\mathrm{c} = \nabla\varphi \times \mathbf{n}_\mathrm{c} \text{ on } \Gamma_\mathrm{c}\}. \end{aligned}$$

For any $\mathbf{w} \in \mathbf{V}$, we multiply the two sides of (2.9) by $\mathbf{w}$ and integrate it in Ω_c to obtain

$$\int_{\Omega_c} \mu\frac{\partial\mathbf{H}_0}{\partial t} \cdot \mathbf{w}dV + \frac{1}{\sigma}\int_{\Omega_c} (\nabla\times\nabla\times\mathbf{H}_0) \cdot \mathbf{w}dV = -\int_{\Omega_c} \mu\frac{\partial\mathbf{H}_s}{\partial t} \cdot \mathbf{w}dV, \tag{2.13}$$

where $\mathbf{w} = \begin{cases} \mathbf{w}, & \text{in } \Omega_c \\ \nabla\varphi, & \text{in } \Omega_e \end{cases}$. Then we can get

$$\int_{\Omega_c} \mu\frac{\partial\mathbf{H}_0}{\partial t} \cdot \mathbf{w}dV + \frac{1}{\sigma}[\int_{\Omega_c} (\nabla\times\mathbf{H}_0)\cdot(\nabla\times\mathbf{w})dV + \int_{\Gamma_c} (\mathbf{n}_c\times(\nabla\times\mathbf{H}_0))\cdot\mathbf{w}dA] = -\int_{\Omega_c} \mu\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{w}dV. \tag{2.14}$$

Let us consider the item $\frac{1}{\sigma}\int_{\Gamma_c}(\mathbf{n}_c \times (\nabla \times \mathbf{H}_0)) \cdot \mathbf{w}dA$. By the Corollary 3.20 of [24] and (2.10), we obtain

$$\begin{aligned} & \tfrac{1}{\sigma}\textstyle\int_{\Gamma_c}(\mathbf{n}_c \times (\nabla\times\mathbf{H}_0)) \cdot \mathbf{w}dA = \tfrac{1}{\sigma}\int_{\Gamma_c}(\nabla\times\mathbf{H}_0)\cdot(\nabla\varphi\times\mathbf{n}_c)dA \\ & = \tfrac{1}{\sigma}\textstyle\int_{\Gamma_c}(\mathbf{n}_c\times(\nabla\times\mathbf{H}_0))\cdot\nabla\varphi dA = -\int_{\Gamma_c}(\mu\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{n}_c)\varphi dA - \int_{\Gamma_c}(\mu\frac{\partial\mathbf{H}_0}{\partial t}\cdot\mathbf{n}_c)\varphi dA \\ & = -\textstyle\int_{\Gamma_c}(\mu\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{n}_c)\varphi dA - \int_{\Gamma_c}\mu\frac{\partial(\nabla\psi\cdot\mathbf{n}_c)}{\partial t}\varphi dA. \end{aligned} \tag{2.15}$$

According to [33], we have

$$-\int_{\Gamma_c}(\nabla\psi\cdot\mathbf{n}_c)\varphi dA=\int_{\Omega_e}\nabla\psi\cdot\nabla\varphi dV,\qquad \forall\,\varphi\in\mathbf{W}_0^1(\Omega_e). \tag{2.16}$$

By (2.14), (2.15) and (2.16), (2.13) can be written as:

$$\begin{aligned}&\int_{\Omega_c}\mu\frac{\partial\mathbf{H}_0}{\partial t}\cdot\mathbf{w}dV+\frac{1}{\sigma}\int_{\Omega_c}(\nabla\times\mathbf{H}_0)\cdot(\nabla\times\mathbf{w})dV+\frac{\partial}{\partial t}\int_{\Omega_e}\mu\nabla\psi\cdot\nabla\varphi dV\\ &=-\int_{\Omega_c}\mu\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{w}dV+\int_{\Gamma_c}(\mu\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{n}_c)\varphi dA.\end{aligned} \tag{2.17}$$

So the initial problem (2.12) has the following variational form:

Find $\mathbf{H}_0\in\mathbf{L}^2((0,T);\mathbf{V})$ such that

$$\begin{cases}\frac{\partial}{\partial t}\int_{\mathbb{R}^3}\mu\mathbf{H}_0\cdot\mathbf{w}dV+\frac{1}{\sigma}\int_{\Omega_c}(\nabla\times\mathbf{H}_0)\cdot(\nabla\times\mathbf{w})dV\\ =-\int_{\Omega_c}\mu\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{w}dV+\int_{\Gamma_c}(\mu\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{n}_c)\varphi dA,\qquad \forall\,\mathbf{w}\in\mathbf{V},\\ \mathbf{H}_0(\cdot,0)=\mathbf{0},\ \psi(\cdot,0)=0.\end{cases} \tag{2.18}$$

By the Galerkin method, it can be proved that the problem (2.18) has a unique solution $\mathbf{H}_0\in\mathbf{L}^2((0,T);\mathbf{V})$, provided that $\mathbf{H}_s$ and $\mathbf{A}_s$ are regular enough.

2.4 Spherical harmonics

In this subsection, we will introduce the spherical harmonics, which are the eigenfunctions of the Laplace-Beltrami operator on the unit sphere. The spherical harmonics will be used to define the natural integral operator for the exterior spherical domain. There are abundant results on the spherical harmonics, see, for example, [20, 26].

Let (x,y,z) be the rectangular coordinates. We adopt the spherical polar coordinates (ρ,θ,ϕ) which satisfy $x=\rho\sin\theta\cos\phi$, $y=\rho\sin\theta\sin\phi$ and $z=\rho\cos\theta$.

The spherical harmonics of order l are the $2l+1$ functions of the form

$$\widehat{Y}_l^m(\theta,\phi)\ =\sqrt{\frac{2l+1}{4\pi}\cdot\frac{(l-m)!}{(l+m)!}}P_l^m(cos\theta)e^{im\phi}, \tag{2.19}$$

where $l=0,1,2,\ldots$ and $m=-l,\ldots,l$, or

$$\begin{cases}Y_{0l}^0(\theta,\phi) &=\sqrt{\frac{2l+1}{4\pi}}P_l^m(cos\theta),\\ Y_{1l}^m(\theta,\phi) &=\sqrt{\frac{2l+1}{2\pi}\cdot\frac{(l-m)!}{(l+m)!}}P_l^m(cos\theta)cos(m\phi),\\ Y_{2l}^m(\theta,\phi) &=\sqrt{\frac{2l+1}{2\pi}\cdot\frac{(l-m)!}{(l+m)!}}P_l^m(cos\theta)sin(m\phi),\end{cases} \tag{2.20}$$

where $l=0,1,2,\ldots$ and $m=1,\ldots,l$.

The associated Legendre functions $P_l^m(x)$ satisfy :

$$\begin{cases}P_l^m(x) &=(-1)^m\frac{1}{2^l l!}(1-x^2)^{\frac{m}{2}}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l, &\text{if } 0\le m\le l,\\ P_l^m(x) &=(-1)^{-m}\frac{(l+m)!}{(l-m)!}P_l^{-m}(x), &\text{if } -l\le m\le 0,\end{cases} \tag{2.21}$$

where $l=0,1,2,\ldots$, $m=-l,\ldots,l$ and $-1\le x\le 1$.

We also have the following orthogonalities

$$\begin{cases} \int_0^{2\pi}\int_0^{\pi} \widehat{Y}_{l'}^{m'}(\theta,\phi)\overline{\widehat{Y}_l^m}(\theta,\phi)\sin\theta d\theta d\phi = \delta_{ll'}\delta_{mm'}, \\ \int_0^{2\pi}\int_0^{\pi} Y_{t'l'}^{m'}(\theta,\phi)Y_{tl}^m(\theta,\phi)\sin\theta d\theta d\phi = \delta_{ll'}\delta_{mm'}\delta_{tt'}, \end{cases}$$

where $t, t' = 0, 1, 2$. Moreover, [26] describes that the two families of spherical harmonics constitute the orthogonal basis of the space $L^2(s)$ respectively, also orthogonal in the space $H^1(s)$, where s denotes the surface of a unit sphere.

3 Coupled variational problem

Let $\{t_0, t_1, \cdots, t_M\}$ be a partition of the time interval $[0, T]$ and $\tau_n = t_n - t_{n-1}$ be the nth length of time segment. In this section, the main purpose is to give the coupled variational formulation of (2.12) at each time step. Above all, we introduce a ball B_R containing the conductor, whose radius is R. The whole unbounded domain Ω_e is decomposed into two subdomains by the surface of the ball. We denote the bounded domain $\Omega_e \cap B_R$ by Ω_{e_1} and the unbounded domain $\Omega_e \backslash \overline{B}_R$ is denoted by Ω_{e_2}. Let the interface $\Gamma_e = \overline{\Omega}_{e_1} \cap \overline{\Omega}_{e_2}$. Then we assume that

$$\mathbf{H}_{0n}(\mathbf{x}) = \mathbf{H}_0(\mathbf{x}, t_n), \quad \psi_{1n}(\mathbf{x}) = \psi_1(\mathbf{x}, t_n), \quad \text{and} \quad \psi_{2n}(\mathbf{x}) = \psi_2(\mathbf{x}, t_n),$$

for $n = 0, \ldots, M$, where $\psi_1 = \psi|_{\Omega_{e_1}}$ and $\psi_2 = \psi|_{\Omega_{e_2}}$. Therefore, we know that

$$\mathbf{H}_{00}(\mathbf{x}) \equiv 0, \quad \psi_{10}(\mathbf{x}) \equiv 0, \quad \text{and} \quad \psi_{20}(\mathbf{x}) \equiv 0.$$

The backward Euler method is applied for the discretization of the time variable. Substituting $\frac{\mathbf{H}_{0n}-\mathbf{H}_{0n-1}}{\tau_n}$ for $\frac{\partial \mathbf{H}_0}{\partial t}|_{t=t_n}$, we have the approximative problem at the time t_n for $n = 1, \ldots, M$:

$$\begin{cases} \mathbf{H}_{0n} + \frac{\tau_n}{\mu\sigma}\nabla\times\nabla\times\mathbf{H}_{0n} = -\tau_n\frac{\partial \mathbf{H}_s}{\partial t}(t_n) + \mathbf{H}_{0n-1}, & \text{in } \Omega_{\mathrm{c}}, \\ \nabla\cdot\mathbf{H}_{0n} = 0, & \text{in } \Omega_{\mathrm{c}}, \\ \triangle\psi_{1n} = 0, & \text{in } \Omega_{\mathrm{e}_1}, \\ \triangle\psi_{2n} = 0, & \text{in } \Omega_{\mathrm{e}_2}, \\ \mathbf{H}_{0n}\times\mathbf{n}_c = \nabla\psi_{1n}\times\mathbf{n}_c, \quad \mathbf{H}_{0n}\cdot\mathbf{n}_c = \nabla\psi_{1n}\cdot\mathbf{n}_c, & \text{on } \Gamma_{\mathrm{c}}, \\ \psi_{1n} = \psi_{2n}, \quad \nabla\psi_{1n}\cdot\mathbf{n}_e = \nabla\psi_{2n}\cdot\mathbf{n}_e, & \text{on } \Gamma_{\mathrm{e}}, \end{cases} \tag{3.1}$$

where $\mathbf{n}_e$ denotes the unit normal vector on Γ_e, which points to the exterior of the B_R. Then we give some notations:

$$\mathbf{V}_1 := \mathbf{H}(\mathbf{curl}, \Omega_c), \quad \mathbf{V}_1^0 := \{\mathbf{u} \in \mathbf{V}_1 \mid \mathbf{u}\times\mathbf{n}_c = 0 \text{ on } \Gamma_{\mathrm{c}}\}, \tag{3.2}$$

$$\widetilde{V}_2 := H^1(\Omega_{e_1}), \qquad \widetilde{V}_2^0 := \{\varphi \in \widetilde{V}_2 \mid \varphi|_{\partial\Omega_{e_1}} = 0\}, \tag{3.3}$$

$$V_2 := H^1(\Omega_{e_1})/P_0, \quad V_2^0 := \{\varphi \in V_2 \mid \varphi|_{\partial\Omega_{e_1}} = [0]\}, \tag{3.4}$$

$$V_3 := \mathbf{W}_0^1(\Omega_{e_2}), \qquad V_3^0 := \{\varphi \in V_3 \mid \varphi = 0 \text{ on } \Gamma_e\}, \tag{3.5}$$

where P_0 is the set of all constants and [0] denotes the element "0" of $H^{\frac{1}{2}}(\partial\Omega_{e_1})/P_0$. We assume

$$\mathbf{f}_n = -\tau_n \frac{\partial \mathbf{H}_s}{\partial t}(t_n) + \mathbf{H}_{0n-1},$$

then the bilinear forms and the functionals associated with (3.1) are

$$a_{n1}(\mathbf{u},\mathbf{v}) = \int_{\Omega_c} \mathbf{u}\cdot\mathbf{v} + \frac{\tau_n}{\mu\sigma}(\nabla\times\mathbf{u})\cdot(\nabla\times\mathbf{v})dV, \quad \forall\ \mathbf{u},\ \mathbf{v}\in\mathbf{V}_1, \tag{3.6}$$

$$a_{n2}(\varphi,\psi) = \int_{\Omega_{e_1}} (\nabla\varphi\cdot\nabla\psi)dV, \quad \forall\ \varphi,\ \psi\in\widetilde{V}_2, \tag{3.7}$$

$$F_{n1}(\mathbf{v}) = \int_{\Omega_c} \mathbf{f}_n\cdot\mathbf{v}dV, \quad \forall\ \mathbf{v}\in\mathbf{V}_1, \tag{3.8}$$

for $n = 1,\ldots,M$.

According to [28], we can get the Poisson integral formula for the exterior spherical domain Ω_{e_2}:

$$\psi_{2n}(r,\theta,\phi) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{(r^2-R^2)\psi_{2n}^d(\theta',\phi')\sin\theta'}{(R^2+r^2-2Rrcos\gamma)^{3/2}}d\theta' d\phi', \quad \text{for r} > \text{R}, \tag{3.9}$$

and the natural boundary integral formula for the exterior spherical domain Ω_{e_2}:

$$\frac{\partial\psi_{2n}}{\partial\mathbf{n}_e}(\theta,\phi) = \sum_{l=0}^{\infty}\frac{(l+1)(2l+1)}{4\pi R}\int_0^{2\pi}\int_0^{\pi} P_l(\cos\gamma)\psi_{2n}^d(\theta',\phi')\sin\theta' d\theta' d\phi', \tag{3.10}$$

where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi)\prime \quad \text{and} \quad \psi_{2n}^d = \psi_{2n}|_{\Gamma_e}.$$

So we obtain the coupled variational form of the problem (3.1):

Find $(\mathbf{H}_{0n},\psi_{1n},\psi_{2n}) \in V_1\times\widetilde{V}_2\times V_3$, $n = 1,\ldots,M$, such that

$$\begin{cases} \psi_{2n}(r,\theta,\phi) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{(r^2-R^2)\psi_{2n}^d(\theta',\phi')\sin\theta'}{(R^2+r^2-2Rrcos\gamma)^{3/2}}d\theta' d\phi', & \text{for r} > \text{R}, \\ a_{n1}(\mathbf{H}_{0n},\mathbf{v}) = F_{n1}(\mathbf{v}), & \forall\ \mathbf{v}\in\mathbf{V}_1^0, \\ \int_{\Omega_c}\mathbf{H}_{0n}\cdot\nabla p dV = 0, & \forall\ p\in H_0^1(\Omega_c), \\ \mathbf{H}_{0n}\times\mathbf{n}_c = \nabla\psi_{1n}\times\mathbf{n}_c, & \text{on } \Gamma_c, \\ \psi_{1n} = \psi_{2n}, & \text{on } \Gamma_e, \\ \int_{\Gamma_e}\frac{\partial\psi_{2n}}{\partial\mathbf{n}_e}dA - \int_{\Gamma_c}(\mathbf{H}_{0n}\cdot\mathbf{n}_c)dA = 0, & \\ a_{n2}(\psi_{1n},\varphi) = \int_{\Gamma_e}\frac{\partial\psi_{2n}}{\partial\mathbf{n}_e}\varphi dA - \int_{\Gamma_c}(\mathbf{H}_{0n}\cdot\mathbf{n}_c)\varphi dA, & \forall\ \varphi\in\widetilde{V}_2. \end{cases} \tag{3.11}$$

4 A solution method

In this section, we introduce some discrete function spaces and the Dirichlet-to-Neumann operator for the exterior Dirichlet problem of the Laplace equation. Then, a backward Euler-DtN alternating algorithm is proposed to solve the discrete problem of (3.11).

4.1 The discretization

To get the discrete form of the problem (3.11), we first give the following discrete function spaces.

The discretizations of $\mathbf{V}_1$ and $\mathbf{V}_1^0$. Let $\{\mathcal{T}_{h_c}^c\}_{h_c>0}$ be a family of tetrahedral meshes within Ω_c, where h_c is the maximum diameter of the meshes. The set of edges of $\mathcal{T}_{h_c}^c$ is denoted by $\mathcal{E}_h$. By using the Nédélec edge elements of the lowest order, we give the finite element space defined on $\mathcal{T}_{h_c}^c$:

$$N_{1,h}^1 := \{\mathbf{v}_h \in H(\mathbf{curl}, \Omega_c) \mid \mathbf{v}_h|_K \in R_1(K), \ \forall\ K \in \mathcal{T}_{h_c}^c\},$$

where $R_1(K)$ is a subset of all linear polynomials on the element K of the form:

$$R_1(K) = \{\mathbf{a} + \mathbf{b} \times \mathbf{x};\ \mathbf{a}\, , \mathbf{b} \in \mathbb{R}^3,\ \mathbf{x} \in K\}.$$

From [18, 25], we know that the tangential components of any edge element function $\mathbf{v}_h$ of $N_{1,h}^1$ are continuous on all edges of every element in $\mathcal{T}_{h_c}^c$ and $\mathbf{v}_h$ is uniquely determined by its moments on edges of $\mathcal{T}_{h_c}^c$:

$$M_e(\mathbf{v}_h) = \{\int_e \mathbf{v}_h \cdot \tau_e ds,\ e \in \mathcal{E}_h\},$$

where τ_e is a unit vector on the edge e. So we can obtain the discretizations of the function spaces $\mathbf{V}_1$ and $\mathbf{V}_1^0$:

$$\mathbf{V}_{1h} := \mathbf{V}_1 \cap N_{1,h}^1, \quad \text{and} \quad \mathbf{V}_{1h}^0 := \mathbf{V}_1^0 \cap N_{1,h}^1.$$

The discretizations of V_2 and $\widetilde{V}_2$. In the domain Ω_{e_1}, we also introduce a family of tetrahedral meshes, $\{\mathcal{T}_{h_e}^e\}_{h_e>0}$, such that the meshes $\mathcal{T}_{h_e}^e$ and $\mathcal{T}_{h_c}^c$ are matched on Γ_c, where h_e is the maximum diameter of the mesh $\mathcal{T}_{h_e}^e$. The set of nodes of $\mathcal{T}_{h_e}^e$ is denoted by $\mathcal{N}_h$. Let $\Omega_{e_1}^h = \bigcup_{K\in\mathcal{T}_{h_e}^e} K$, which is an inscribed polyhedron of Ω_{e_1}, then we use the piecewise linear finite elements to get the finite element space defined on $\mathcal{T}_{h_e}^e$:

$$S_h := \{\varphi_h \in H^1(\Omega_{e_1}^h) \mid \varphi_h|_K \in P_1(K), \forall\ K \in \mathcal{T}_{h_e}^e\},$$

where $P_1(K)$ denotes the set of polynomials of total degree at most 1 defined on the element K. For any $\varphi_h \in S_h$, it can be uniquely determined by

$$M_a(\varphi_h) = \{\varphi_h(a),\ a \in \mathcal{N}_h\}.$$

Since Ω_{e_1} is a curvilinear domain, we need curvilinear tetrahedral mesh near Γ_e to subdivide Ω_{e_1} completely. The following map $\tilde{F}_k$ described in [24] will be used to obtain the curvilinear tetrahedral mesh of Ω_{e_1}.

Let $K \in \mathcal{T}_{h_e}^e$ be a tetrahedron, which has two or three vertices on Γ_e, with barycentric coordinate functions λ_j, $1 \le j \le 4$. We can give Dubois' definition of $\tilde{F}_k : K \to \tilde{K}$, where $\tilde{K}$ denotes the curvilinear tetrahedron having the same vertices with K [21].

1. if K and $\tilde{K}$ have two common vertices on Γ_e , $\mathbf{a}_1$ and $\mathbf{a}_2$, we define

$$\tilde{F}_k(\mathbf{x}) \quad = (1 - \lambda_3 - \lambda_4) P_{\Gamma_e}(\tfrac{\lambda_1\mathbf{a}_1+\lambda_2\mathbf{a}_2}{\lambda_1+\lambda_2}) + \lambda_3\mathbf{a}_3 + \lambda_4\mathbf{a}_4, \tag{4.1}$$

2. if K and $\tilde{K}$ have three common vertices on Γ_e , $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, we define

$$\tilde{F}_k(\mathbf{x}) \quad = (1-\lambda_4)P_{\Gamma_e}(\tfrac{\lambda_1\mathbf{a}_1+\lambda_2\mathbf{a}_2+\lambda_3\mathbf{a}_3}{\lambda_1+\lambda_2+\lambda_3}) + \lambda_4\mathbf{a}_4, \tag{4.2}$$

where

$$P_{\Gamma_e}\mathbf{x} \quad = R\mathbf{x}/|\mathbf{x}|, \qquad \mathbf{x} \in \Omega_{\Gamma_e}. \tag{4.3}$$

$\Omega_{\Gamma_e} \subset \Omega_{e_1}$ is a neighborhood of Γ_e and P_{Γ_e} just projects the points in Ω_{Γ_e} normally onto Γ_e, which is well defined provided $\mathbf{x} \neq 0$. It is easy to see that $\tilde{F}_k$ is a continuously differentiable, invertible and surjective mapping.

Let

$$B_{h_e} = \{K \in \mathcal{T}^e_{h_e} \mid \; K \text{ has two or three vertices on } \Gamma_e\},$$

then we can define $\widehat{F}_h : \Omega^h_{e_1} \to \Omega_{e_1}$,

$$\widehat{F}_h(\mathbf{x}) = \begin{cases} \tilde{F}_k(\mathbf{x}), & \text{if } \mathbf{x} \in K \in B_{h_e}, \\ \mathbf{x}, & \text{if } \mathbf{x} \in K \in \mathcal{T}^e_{h_e} \backslash B_{h_e}. \end{cases}$$

The discrete function space of $\widetilde{V}_2$ is obtained as follows:

$$\widetilde{V}_{2h} := \{\widetilde{\varphi}_h \in H^1(\Omega_{e_1}) \mid \widetilde{\varphi}_h \circ \widehat{F}_h = \varphi_h \text{ for some } \varphi_h \in S_h\},$$

then we can get the discretizations of the function spaces V_2, V_2^0 and $\widetilde{V}_2^0$:

$$V_{2h} = V_2 \cap (\widetilde{V}_{2h}/P_0), \quad V^0_{2h} = V^0_2 \cap (\widetilde{V}_{2h}/P_0) \quad \text{and} \quad \widetilde{V}^0_{2h} = \widetilde{V}^0_2 \cap \widetilde{V}_{2h}.$$

Since the curvilinear tetrahedral mesh of Ω_{e_1} has created a curvilinear triangulation on the boundary $\partial\Omega_{e_1}$, we can get the discrete spaces of $H^{\frac{1}{2}}(\partial\Omega_{e_1})$:

$$\chi_h := \{\widetilde{\varphi}_h|_{\partial\Omega_{e_1}} \mid \widetilde{\varphi}_h \in \widetilde{V}_{2h}\}.$$

4.2 Dirichlet-to-Neumann operator

For the exterior Dirichlet problem of the Laplace equation, we will give Dirichlet-to-Neumann operator which is actually the natural integral operator discussed in [29, 31].

According to [26], the spherical harmonics $\widehat{Y}_l^m$ form a basis of the linear space $L^2(s)$, so any function $\varphi_d \in L^2(\Gamma_e)$ can be expanded as a sum of spherical harmonics $\widehat{Y}_l^m$:

$$\begin{cases} \varphi_d(\theta,\phi) = & \sum\limits_{l=0}^{\infty}\sum\limits_{m=-l}^{l} \widehat{\varphi}_l^m \widehat{Y}_l^m(\theta,\phi), \\ \widehat{\varphi}_l^m = & \frac{1}{R^2}\int_{\Gamma_e} \varphi_d(\theta,\phi)\overline{\widehat{Y}_l^m}(\theta,\phi)dA. \end{cases} \tag{4.4}$$

φ_d can also be expanded as a sum of the other family of spherical harmonics (2.20):

$$\begin{cases} \varphi_d(\theta,\phi) = & \sum\limits_{l=0}^{\infty}\left(\sum\limits_{m=1}^{l}(\varphi_{1l}^m Y_{1l}^m(\theta,\phi) + \varphi_{2l}^m Y_{2l}^m(\theta,\phi)) + \varphi_{0l}^0 Y_{0l}^0(\theta,\phi)\right), \\ \varphi_{0l}^0 = & \frac{1}{R^2}\int_{\Gamma_e}\varphi_d(\theta,\phi)Y_{0l}^0(\theta,\phi)dA, \\ \varphi_{1l}^m = & \frac{1}{R^2}\int_{\Gamma_e}\varphi_d(\theta,\phi)Y_{1l}^m(\theta,\phi)dA, \\ \varphi_{2l}^m = & \frac{1}{R^2}\int_{\Gamma_e}\varphi_d(\theta,\phi)Y_{2l}^m(\theta,\phi)dA. \end{cases} \tag{4.5}$$

For $s>0$, the space $H^s(\Gamma_e)$ is constituted of the functions in $L^2(\Gamma_e)$ such that the series

$$\|\varphi_d\|^2_{H^s(\Gamma_e)} := R^2 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l+1)^{2s}|\widehat{\varphi}_l^m|^2 = R^2\sum_{l=0}^{\infty}(l+1)^{2s}\left(\sum_{m=1}^{l}(|\varphi_{1l}^m|^2+|\varphi_{2l}^m|^2)+|\varphi_{0l}^0|^2\right)$$

is convergent. Its hermitian product is

$$\langle\varphi_d,\psi_d\rangle_{H^s(\Gamma_e)} := R^2 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l+1)^{2s}\widehat{\varphi}_l^m\overline{\widehat{\psi}_l^m} = R^2\sum_{l=0}^{\infty}(l+1)^{2s}\left(\sum_{m=1}^{l}(\varphi_{1l}^m\psi_{1l}^m+\varphi_{2l}^m\psi_{2l}^m)+\varphi_{0l}^0\psi_{0l}^0\right).$$

For $s<0$, the space $H^s(\Gamma_e)$ is the space of distributions in $\mathcal{D}'(\Gamma_e)$ such that the series

$$\|\varphi_d\|^2_{H^s(\Gamma_e)} := R^2 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l+1)^{2s}|\widehat{\varphi}_l^m|^2 = R^2\sum_{l=0}^{\infty}(l+1)^{2s}\left(\sum_{m=1}^{l}(|\varphi_{1l}^m|^2+|\varphi_{2l}^m|^2)+|\varphi_{0l}^0|^2\right)$$

is convergent.

Then we introduce two families of harmonic functions

$$\widehat{K}_l^m(r,\theta,\phi) = \frac{1}{r^{l+1}}\widehat{Y}_l^m(\theta,\phi), \tag{4.6}$$

for $l=0,1,2,\dots$ and $m=-l,\dots,l$, and

$$\begin{cases} K_{0l}^0(r,\theta,\phi) = \frac{1}{r^{l+1}}Y_{0l}^0(\theta,\phi),\\ K_{1l}^m(r,\theta,\phi) = \frac{1}{r^{l+1}}Y_{1l}^m(\theta,\phi),\\ K_{2l}^m(r,\theta,\phi) = \frac{1}{r^{l+1}}Y_{2l}^m(\theta,\phi),\end{cases} \tag{4.7}$$

for $l=0,1,2,\dots$ and $m=1,2,\dots,l$, which are not smooth at the origin and tend to zero at infinity.

Consider the exterior Dirichlet problem

$$\begin{cases} \triangle\psi_{e_2}=0, & \text{in } \Omega_{e_2},\\ \psi_{e_2}=\psi_d, & \text{on } \Gamma_e.\end{cases} \tag{4.8}$$

Since ψ_d can be expanded as

$$\psi_d(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\widehat{\psi}_l^m\widehat{Y}_l^m(\theta,\phi) = \sum_{l=0}^{\infty}\left(\sum_{m=1}^{l}(\psi_{1l}^mY_{1l}^m(\theta,\phi)+\psi_{2l}^mY_{2l}^m(\theta,\phi))+\psi_{0l}^0Y_{0l}^0(\theta,\phi)\right), \tag{4.9}$$

according to [26], we can obtain

$$\psi_{e_2}(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}R^{l+1}\widehat{\psi}_l^m\widehat{Y}_l^m(\theta,\phi)\frac{1}{r^{l+1}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}R^{l+1}\widehat{\psi}_l^m\widehat{K}_l^m(r,\theta,\phi), \quad \text{for } r>R, \tag{4.10}$$

and

$$\begin{aligned}\psi_{e_2}(r,\theta,\phi) &= \sum_{l=0}^{\infty}\frac{R^{l+1}}{r^{l+1}}\left(\sum_{m=1}^{l}(\psi_{1l}^mY_{1l}^m(\theta,\phi)+\psi_{2l}^mY_{2l}^m(\theta,\phi))+\psi_{0l}^0Y_{0l}^0(\theta,\phi)\right)\\ &= \sum_{l=0}^{\infty}R^{l+1}\left(\sum_{m=1}^{l}(\psi_{1l}^mK_{1l}^m(r,\theta,\phi)+\psi_{2l}^mK_{2l}^m(r,\theta,\phi))+\psi_{0l}^0K_{0l}^0(r,\theta,\phi)\right), \quad \text{for } r>R.\end{aligned} \tag{4.11}$$

Since it is known that

$$\left.\frac{\partial\widehat{K}_l^m}{\partial\mathbf{n}_e}\right|_{\Gamma_e} = -(l+1)\frac{1}{R^{l+2}}\widehat{Y}_l^m,$$

we can define the Dirichlet-to-Neumann operator $\mathcal{G}_e : H^{\frac{1}{2}}(\Gamma_e) \to H^{-\frac{1}{2}}(\Gamma_e)$, which associates ψ_d with the normal derivative $\frac{\partial \psi_d}{\partial \mathbf{n}_e}$ of the exterior Dirichlet problem (4.8). For ψ_d given by (4.9), $\mathcal{G}_e$ satisfies that

$$\begin{aligned} \mathcal{G}_e\psi_d &= -\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \tfrac{1}{R}(l+1)\widehat{\psi}_l^m \widehat{Y}_l^m(\theta,\phi) \\ &= -\sum_{l=0}^{\infty} \tfrac{1}{R}(l+1)\left(\sum_{m=1}^{l}(\psi_{1l}^m Y_{1l}^m(\theta,\phi) + \psi_{2l}^m Y_{2l}^m(\theta,\phi)) + \psi_{0l}^0 Y_{0l}^0(\theta,\phi)\right) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \langle \mathcal{G}_e\psi_d, \varphi_d\rangle_{\Gamma_e} &= \textstyle\int_{\Gamma_e}(\mathcal{G}_e\psi_d)\varphi_d dA \\ &= -\sum_{l=0}^{\infty} R(l+1)\left(\sum_{m=1}^{l}(\psi_{1l}^m\varphi_{1l}^m + \psi_{2l}^m\varphi_{2l}^m) + \psi_{0l}^0\varphi_{0l}^0\right), \quad \forall \varphi_d \in H^{\frac{1}{2}}(\Gamma_e), \end{aligned} \tag{4.13}$$

where φ_d has the expansion

$$\varphi_d(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \widehat{\varphi}_l^m \widehat{Y}_l^m(\theta,\phi) = \sum_{l=0}^{\infty}\left(\sum_{m=1}^{l}(\varphi_{1l}^m Y_{1l}^m(\theta,\phi) + \varphi_{2l}^m Y_{2l}^m(\theta,\phi)) + \varphi_{0l}^0 Y_{0l}^0(\theta,\phi)\right).$$

4.3 Backward Euler-DtN alternating algorithm

Based on the previous preparations, we will propose a backward Euler-DtN alternating algorithm for solving the discrete problem of (3.11) in this subsection. Since our ultimate aim is to obtain $\mathbf{H}_0 \in \mathbf{V}$, we only need to get $\nabla\psi_1$ instead of ψ_1 in Ω_{e_1}. In the algorithm, we only obtain $\widehat{\psi}_{1n} \in V_{2h}$ such that $\nabla\widehat{\psi}_{1n} = \nabla\psi_{1n}$ for $n = 1, \dots, M$. For ease of notation, we denote $\widehat{\psi}_{1n}$ by ψ_{1n} in the rest of this paper. Let

$$\mathcal{Z} := \{\lambda \mid \lambda \in \chi_h/P_0\},$$

then the backward Euler-DtN alternating algorithm is described as follows:

1. Let $n := 1$ and $\widetilde{\mathbf{H}}_{00}^h = \mathbf{H}_{00} \equiv 0$.
2. Solve the discrete problem of (3.11) at t_n:
   - (a) Given the initial value $\lambda_{n,0} \in \mathcal{Z}$. Let $m := 0$.
   - (b) Let
     $$\widetilde{F}_{n1}(\mathbf{v}) = \int_{\Omega_c}(-\tau_n\frac{\partial \mathbf{H}_s}{\partial t}(t_n) + \widetilde{\mathbf{H}}_{0n-1}^h)\cdot \mathbf{v}dV, \quad \forall\, \mathbf{v} \in \mathbf{V}_{1h}.$$
     Find $\mathbf{H}_{0n,m}^h \in \mathbf{V}_{1h}$ such that
     $$\begin{cases} a_{n1}(\mathbf{H}_{0n,m}^h, \mathbf{v}) = \widetilde{F}_{n1}(\mathbf{v}), & \forall\, \mathbf{v} \in \mathbf{V}_{1h}^0, \\ \mathbf{H}_{0n,m}^h \times \mathbf{n}_c = \nabla\lambda_{n,m} \times \mathbf{n}_c, & \text{on } \Gamma_c. \end{cases} \tag{4.14}$$
   - (c) Expand $\lambda_{n,m}|_{\Gamma_e}$ in the series form
     $$\lambda_{n,m}|_{\Gamma_e} = c + \sum_{l=1}^{\infty}\left(\sum_{k=1}^{l}(q_{n,1l}^{m,k} Y_{1l}^k + q_{n,2l}^{m,k} Y_{2l}^k) + q_{n,0l}^{m,0} Y_{0l}^0\right), \tag{4.15}$$

where c can be any real number. Then we define $\lambda^e_{n,m} \in \chi_h$ such that

$$\lambda^e_{n,m}|_{\Gamma_e} = -\frac{1}{4\pi R}\int_{\Gamma_c} \mathbf{H}^h_{0n,m}\cdot \mathbf{n}_c dA + \sum_{l=1}^{\infty}\left(\sum_{k=1}^{l}(q^{m,k}_{n,1l}Y^k_{1l} + q^{m,k}_{n,2l}Y^k_{2l}) + q^{m,0}_{n,0l}Y^0_{0l}\right), \quad (4.16)$$

and $\lambda_{n,m} = [\lambda^e_{n,m}]$.

(d) Take $\psi^h_{2n,m}|_{\Gamma_e} = \lambda^e_{n,m}|_{\Gamma_e}$, then we can get $\psi^h_{2n,m}$ and $\frac{\partial \psi^h_{2n,m}}{\partial \mathbf{n}_e}$ by using (4.11) and (4.12).

$$\begin{aligned}\psi^h_{2n,m}(r,\theta,\phi) &= -\tfrac{1}{4\pi r}\textstyle\int_{\Gamma_c} \mathbf{H}^h_{0n,m}\cdot \mathbf{n}_c dA \\ &\quad + \sum_{l=1}^{\infty} R^{l+1}\left(\sum_{m=1}^{l}(q^{m,k}_{n,1l}K^m_{1l} + q^{m,k}_{n,2l}K^m_{2l}) + q^{m,0}_{n,0l}K^0_{0l}\right), \quad \text{for } r > R.\end{aligned}$$

$$\begin{aligned}\frac{\partial \psi^h_{2n,m}}{\partial \mathbf{n}_e} &= \mathcal{G}_e(\lambda^e_{n,m}|_{\Gamma_e}) \\ &= \tfrac{1}{4\pi R^2}\textstyle\int_{\Gamma_c} \mathbf{H}^h_{0n,m}\cdot \mathbf{n}_c dA - \sum_{l=1}^{\infty}\frac{(l+1)}{R}\left(\sum_{k=1}^{l}(q^{m,k}_{n,1l}Y^k_{1l} + q^{m,k}_{n,2l}Y^k_{2l}) + q^{m,0}_{n,0l}Y^0_{0l}\right).\end{aligned}$$

(e) Find $\psi^h_{1n,m} \in V_{2h}$ such that

$$\int_{\Omega_{e_1}} \nabla\psi^h_{1n,m}\cdot\nabla\varphi dV = \int_{\Gamma_e}\frac{\partial \psi^h_{2n,m}}{\partial \mathbf{n}_e}\varphi dA - \int_{\Gamma_c}(\mathbf{H}^h_{0n,m}\cdot \mathbf{n}_c)\varphi dA, \qquad \forall\, \varphi \in V_{2h}. \quad (4.17)$$

(f) If the accuracy of the approximation is enough, then iteration stops and let

$$\widetilde{\mathbf{H}}^h_{0n} = \begin{cases} \mathbf{H}^h_{0n,m}, & \text{in } \Omega_c \\ \nabla\psi^h_{1n,m}, & \text{in } \Omega_{e_1} \\ \nabla\psi^h_{2n,m}, & \text{in } \Omega_{e_2} \end{cases};$$

else

$$\lambda_{n,m+1} = (1-\vartheta_{n,m})\lambda_{n,m} + \vartheta_{n,m}\psi^h_{1n,m}|_{\partial\Omega_{e_1}},$$

where $\vartheta_{n,m}$ is the mth linear relaxation factor at the time t_n.

(g) Let $m := m+1$, go to the step (b).

3. If $n < M$, then $n := n+1$ and go to step 2.

Remark 4.1. *The aim of choosing $\lambda^e_{n,m}$ in (c) of step 2 is to ensure the consistency of the problem (4.17).*

5 Convergence of backward Euler-DtN alternating algorithm

5.1 Preconditioned Richardson iterative method

In order to give the convergence of the backward Euler-DtN alternating algorithm, we first need to get the convergence of the DtN alternating algorithm at each time step.

For any $\lambda_n \in \mathcal{Z}$, we introduce the following discrete problems at t_n for $n = 1, \ldots, M$.

Find $\mathbf{R}_n^H\lambda_n \in \mathbf{V}_{1h}$ such that

$$\begin{cases} a_{n1}(\mathbf{R}_n^H\lambda_n, \mathbf{v}) = 0, & \forall\ \mathbf{v} \in\ \mathbf{V}_{1h}^0, \\ \mathbf{R}_n^H\lambda_n \times \mathbf{n}_c = \nabla\lambda_n \times \mathbf{n}_c, & \text{on } \Gamma_c. \end{cases} \tag{5.1}$$

Find $\mathbf{T}_n^H\widetilde{\mathbf{f}}_n \in \mathbf{V}_{1h}^0$ such that

$$a_{n1}(\mathbf{T}_n^H\widetilde{\mathbf{f}}_n, \mathbf{v}) = \widetilde{F}_{n1}(\mathbf{v}), \qquad \forall\ \mathbf{v} \in\ \mathbf{V}_{1h}^0, \tag{5.2}$$

where

$$\widetilde{\mathbf{f}}_n = -\tau_n\frac{\partial \mathbf{H}_s}{\partial t}(t_n) + \widetilde{\mathbf{H}}_{0n-1}^h \text{ and } \widetilde{F}_{n1}(\mathbf{v}) = \int_{\Omega_c} \widetilde{\mathbf{f}}_n \cdot \mathbf{v} dV, \quad \forall\ \mathbf{v} \in \mathbf{V}_{1h}.$$

Find $\mathbf{H}_{0n}^{c,h} \in \mathbf{V}_{1h}$ such that

$$\begin{cases} a_{n1}(\mathbf{H}_{0n}^{c,h}, \mathbf{v}) = \widetilde{F}_{n1}(\mathbf{v}), & \forall\ \mathbf{v} \in\ \mathbf{V}_{1h}^0, \\ \mathbf{H}_{0n}^{c,h} \times \mathbf{n}_c = \nabla\lambda_n \times \mathbf{n}_c, & \text{on } \Gamma_c. \end{cases} \tag{5.3}$$

Find $R_n^{\psi_{e_1}}\lambda_n \in V_{2h}$ such that

$$\begin{cases} \int_{\Omega_{e_1}} \nabla R_n^{\psi_{e_1}}\lambda_n \cdot \nabla\varphi dV = 0, & \forall\ \varphi \in\ V_{2h}^0, \\ R_n^{\psi_{e_1}}\lambda_n = \lambda_n, & \text{on } \partial\Omega_{e_1}. \end{cases} \tag{5.4}$$

From the definitions of the discrete problems, it is known that

$$\mathbf{H}_{0n}^{c,h} = \mathbf{R}_n^H\lambda_n + \mathbf{T}_n^H\widetilde{\mathbf{f}}_n.$$

We will define the discrete Steklov-Poincaré operators S_{1n}^h, S_{2n}^h, S_{3n}^h and the operator Φ_n^h at t_n for $n = 1, \ldots, M$.

$$\langle S_{1n}^h\lambda_{1n}, \lambda_{2n}\rangle = \int_{\Gamma_c} \mathbf{R}_n^H\lambda_{1n} \cdot \mathbf{n}_c\lambda_{2n} dA, \qquad \forall\ \lambda_{1n}, \lambda_{2n} \in\ \mathcal{Z}. \tag{5.5}$$

$$\langle S_{2n}^h\lambda_{1n}, \lambda_{2n}\rangle = \int_{\Omega_{e_1}} \nabla R_n^{\psi_{e_1}}\lambda_{1n} \cdot \nabla R_n^{\psi_{e_1}}\lambda_{2n} dV, \qquad \forall\ \lambda_{1n}, \lambda_{2n} \in\ \mathcal{Z}. \tag{5.6}$$

For any $\lambda_{1n}, \lambda_{2n} \in \mathcal{Z}$, we have

$$\lambda_{in}|_{\Gamma_e} = c + \sum_{l=1}^{\infty}\left(\sum_{k=1}^{l}(q_{in,1l}^k Y_{1l}^k + q_{in,2l}^k Y_{2l}^k) + q_{in,0l}^0 Y_{0l}^0\right), \quad \text{for } i = 1, 2, \tag{5.7}$$

where c can be any real number. Then λ_{2n}^0 can be defined. it satisfies that

$$\lambda_{2n}^0|_{\Gamma_e} = \sum_{l=1}^{\infty}\left(\sum_{k=1}^{l}(q_{2n,1l}^k Y_{1l}^k + q_{2n,2l}^k Y_{2l}^k) + q_{2n,0l}^0 Y_{0l}^0\right),$$

and $\lambda_{2n} = [\lambda_{2n}^0]$.

Let

$$\mathbf{H}_{01n}^{c,h} = \mathbf{R}_n^H\lambda_{1n} + \mathbf{T}_n^H\widetilde{\mathbf{f}}_n$$

and define $\lambda_{1n}^e \in \chi_h$ such that

$$\lambda_{1n}^e|_{\Gamma_e} = -\frac{1}{4\pi R}\int_{\Gamma_c} \mathbf{H}_{01n}^{c,h} \cdot \mathbf{n}_c dA + \sum_{l=1}^{\infty}\left(\sum_{k=1}^{l}(q_{1n,1l}^k Y_{1l}^k + q_{1n,2l}^k Y_{2l}^k) + q_{1n,0l}^0 Y_{0l}^0\right), \tag{5.8}$$

and $\lambda_{1n} = [\lambda_{1n}^e]$.

we can define

$$\begin{aligned} \langle S_{3n}^h \lambda_{1n}, \lambda_{2n} \rangle = & -\int_{\Gamma_e} \frac{\partial \lambda_{1n}^e}{\partial \mathbf{n}_e} \lambda_{2n}^0 dA \\ = & \sum_{l=1}^{\infty} \frac{(l+1)}{R} \left(\sum_{k=1}^{l} (q_{1n,1l}^k q_{2n,1l}^k + q_{1n,2l}^k q_{2n,2l}^k) + q_{1n,0l}^0 q_{2n,0l}^0 \right), \quad \forall\ \lambda_{1n}, \lambda_{2n} \in\ \mathcal{Z}, \end{aligned} \tag{5.9}$$

$$\langle \Phi_n^h, \lambda_{2n} \rangle = -\int_{\Gamma_c} (\mathbf{T}_n^H \widetilde{\mathbf{f}}_n \cdot \mathbf{n}_c) \lambda_{2n}^0 dA, \qquad \forall\ \lambda_{2n} \in\ \mathcal{Z}, \tag{5.10}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{Z}'$ and $\mathcal{Z}$.

Let

$$S_n^h = S_{1n}^h + S_{2n}^h + S_{3n}^h, \text{ for } n = 1, 2, \dots, M,$$

then we introduce the following discrete problems:

Find $\lambda_n^\dagger \in \mathcal{Z}$ such that

$$\langle S_n^h \lambda_n^\dagger, \eta \rangle = \langle \Phi_n^h, \eta \rangle, \qquad \forall\ \eta \in\ \mathcal{Z}, \text{ for n} = 1, \dots, \text{M}. \tag{5.11}$$

It is easy to see that the operator S_{2n}^h is symmetric and positive definite in $\mathcal{Z}$ for $n = 1, 2, \dots, M$, hence we can apply the Richardson method with S_{2n}^h as a preconditioner for solving (5.11) at t_n, i.e.

given $\lambda_{n,0} \in \mathcal{Z}$, for each $m \geq 0$, solve

$$\begin{aligned} \lambda_{n,m+1} &= \lambda_{n,m} + \vartheta_{n,m} (S_{2n}^h)^{-1} [\Phi_n^h - S_n^h \lambda_{n,m}] \\ &= (1 - \vartheta_{n,m}) \lambda_{n,m} + \vartheta_{n,m} (S_{2n}^h)^{-1} [\Phi_n^h - (S_{1n}^h + S_{3n}^h) \lambda_{n,m}], \end{aligned} \tag{5.12}$$

where $\vartheta_{n,m}$ is the mth linear relaxation factor at t_n.

In the following, we will give the equivalence of the Richardson method (5.12) and the DtN alternating method proposed in the second step of the backward Euler-DtN alternating algorithm at each time step.

Lemma 5.1. *The DtN alternating method proposed in the second step of the backward Euler-DtN alternating algorithm is equivalent to the preconditioned Richardson iterative method (5.12) at t_n for $n = 1, \dots, M$.*

Proof: We first define $\mathbf{H}_{0n}^{\dagger h}$, $\psi_{1n}^{\dagger h}$ and $\psi_{2n}^{\dagger h}$.

Find $\mathbf{H}_{0n}^{\dagger h} \in \mathbf{V}_{1h}$ such that

$$\begin{cases} a_{n1}(\mathbf{H}_{0n}^{\dagger h}, \mathbf{v}) = \widetilde{F}_{n1}(\mathbf{v}), & \forall\ \mathbf{v} \in \mathbf{V}_{1h}^0, \\ \mathbf{H}_{0n}^{\dagger h} \times \mathbf{n}_c = \nabla \lambda_n^\dagger \times \mathbf{n}_c, & \text{on } \Gamma_c. \end{cases}$$

Expand $\lambda_n^\dagger|_{\Gamma_e}$ in the series form

$$\lambda_n^\dagger|_{\Gamma_e} = c + \sum_{l=1}^{\infty} \left(\sum_{k=1}^{l} (q_{n,1l}^{\dagger,k} Y_{1l}^k + q_{n,2l}^{\dagger,k} Y_{2l}^k) + q_{n,0l}^{\dagger,0} Y_{0l}^0 \right),$$

where c can be any real number. Then we define $\lambda_n^e \in \chi_h$ satisfying

$$\lambda_n^e|_{\Gamma_e} = -\frac{1}{4\pi R}\int_{\Gamma_c} \mathbf{H}_{0n}^{\dagger h}\cdot \mathbf{n}_c dA + \sum_{l=1}^{\infty}\left(\sum_{k=1}^{l}(q_{n,1l}^{\dagger,k}Y_{1l}^k + q_{n,2l}^{\dagger,k}Y_{2l}^k) + q_{n,0l}^{\dagger,0}Y_{0l}^0\right),$$

and $\lambda_n^\dagger = [\lambda_n^e]$.

By using (4.11), (4.12) and $\psi_{2n}^{\dagger h}|_{\Gamma_e} = \lambda_n^e|_{\Gamma_e}$, we can obtain $\psi_{2n}^{\dagger h}$ and $\frac{\partial \psi_{2n}^{\dagger h}}{\partial \mathbf{n}_e}|_{\Gamma_e}$.

Find $\psi_{1n}^{\dagger h} \in V_{2h}$ such that

$$\int_{\Omega_{e_1}} \nabla\psi_{1n}^{\dagger h}\cdot\nabla\varphi dV = \int_{\Gamma_e}\frac{\partial \psi_{2n}^{\dagger h}}{\partial \mathbf{n}_e}\varphi dA - \int_{\Gamma_c}(\mathbf{H}_{0n}^{\dagger h}\cdot\mathbf{n}_c)\varphi dA, \qquad \forall\ \varphi\in V_{2h}.$$

Define

$$\mathbf{e}_{Hn,m}^h := \mathbf{H}_{0n}^{\dagger h} - \mathbf{H}_{0n,m}^h, \quad e_{\psi_2 n,m}^h := \psi_{2n}^{\dagger h} - \psi_{2n,m}^h, \quad e_{\psi_1 n,m}^h := \psi_{1n}^{\dagger h} - \psi_{1n,m}^h, \quad \mathbf{e}_{\lambda n,m}^h := \lambda_n^\dagger - \lambda_{n,m}.$$

We can see that

$$\begin{cases} a_{n1}(\mathbf{e}_{Hn,m}^h, \mathbf{v}) = 0, & \forall\ \mathbf{v}\in\ \mathbf{V}_{1h}^0,\\ \mathbf{e}_{Hn,m}^h\times\mathbf{n}_c = \nabla\mathbf{e}_{\lambda n,m}^h|_{\Gamma_c}\times\mathbf{n}_c, & \text{on } \Gamma_\mathrm{c}, \end{cases}$$

$$\int_{\Omega_{e_1}} \nabla e_{\psi_1 n,m}^h\cdot\nabla\varphi dV = \int_{\Gamma_e}\frac{\partial e_{\psi_2 n,m}^h}{\partial \mathbf{n}_e}\varphi dA - \int_{\Gamma_c}(\mathbf{e}_{Hn,m}^h\cdot\mathbf{n}_c)\varphi dA, \qquad \forall\ \varphi\in V_{2h},$$

and

$$\mathbf{e}_{\lambda n,m+1}^h = \vartheta_{n,m}\eta_{n,m} + (1-\vartheta_{n,m})\mathbf{e}_{\lambda n,m}^h,$$

where $\eta_{n,m} = e_{\psi_1 n,m}^h|_{\partial\Omega_{e_1}}$.

Since

$$\langle S_{2n}^h\eta_{n,m},\zeta\rangle = -\left(\langle S_{1n}^h\mathbf{e}_{\lambda n,m}^h,\zeta\rangle + \langle S_{3n}^h\mathbf{e}_{\lambda n,m}^h,\zeta\rangle\right),\ \forall\ \zeta\in\mathcal{Z},$$

we have

$$S_{2n}^h\eta_{n,m} = -(S_{1n}^h\mathbf{e}_{\lambda n,m}^h + S_{3n}^h\mathbf{e}_{\lambda n,m}^h).$$

Moreover, it is known that

$$\begin{aligned} \mathbf{e}_{\lambda n,m+1}^h - \mathbf{e}_{\lambda n,m}^h &= \vartheta_{n,m}\eta_{n,m} + (1-\vartheta_{n,m})\mathbf{e}_{\lambda n,m}^h - \mathbf{e}_{\lambda n,m}^h\\ &= \vartheta_{n,m}\eta_{n,m} - \vartheta_{n,m}\mathbf{e}_{\lambda n,m}^h. \end{aligned}$$

So

$$S_{2n}^h(\mathbf{e}_{\lambda n,m+1}^h - \mathbf{e}_{\lambda n,m}^h) = -\vartheta_{n,m}(S_{1n}^h + S_{3n}^h)\mathbf{e}_{\lambda n,m}^h - \vartheta_{n,m}S_{2n}^h\mathbf{e}_{\lambda n,m}^h.$$

Then we can get

$$S_{2n}^h(\lambda_{n,m} - \lambda_{n,m+1}) = -\vartheta_{n,m}(S_{1n}^h + S_{3n}^h)(\lambda_n^\dagger - \lambda_{n,m}) - \vartheta_{n,m}S_{2n}^h(\lambda_n^\dagger - \lambda_{n,m}).$$

We notice that

$$(S_{1n}^h + S_{2n}^h + S_{3n}^h)\lambda_n^\dagger = \Phi_n^h,$$

therefore,

$$S^h_{2n}\lambda_{n,m+1} = S^h_{2n}\lambda_{n,m} + \vartheta_{n,m}[\Phi^h_n - (S^h_{1n} + S^h_{2n} + S^h_{3n})\lambda_{n,m}].$$

So the DtN alternating algorithm at t_n is equivalent to the preconditioned Richardson iterative method:

Given $\lambda_{n,0} \in \mathcal{Z}$, for each $m \geq 0$, solve

$$\begin{aligned}\lambda_{n,m+1} &= \lambda_{n,m} + \vartheta_{n,m}(S^h_{2n})^{-1}[\Phi^h_n - S^h_n\lambda_{n,m}]\\ &= (1-\vartheta_{n,m})\lambda_{n,m} + \vartheta_{n,m}(S^h_{2n})^{-1}[\Phi^h_n - (S^h_{1n} + S^h_{3n})\lambda_{n,m}], \quad \text{for } n = 1, 2, \ldots, M. \qquad \square\end{aligned}$$

5.2 Convergence of the preconditioned Richardson iterative method

In order to obtain the convergence of the DtN alternating algorithm at each time step, according to the equivalence given in lemma 5.1, we will devote ourselves to getting the convergence of the preconditioned Richardson iterative method (5.12) at t_n for $n = 1, 2, \ldots, M$.

We first present some auxiliary lemmata.

Lemma 5.2. *The operator $S^h_{1n} : \mathcal{Z} \to \mathcal{Z}'$ is symmetric and positive semi-definite for $n = 1, \ldots, M$.*

Proof: For any $\lambda_{1n},\ \lambda_{2n} \in \mathcal{Z}$, according to the definition of the operator S^h_{1n}, we have

$$\langle S^h_{1n}\lambda_{1n}, \lambda_{2n}\rangle = \int_{\Gamma_c} \mathbf{R}^H_n\lambda_{1n} \cdot \mathbf{n}_c\lambda_{2n}dA, \qquad \text{for } n = 1, \ldots, M.$$

Let V_{ch} be the piecewise linear finite element space of $H^1(\Omega_c)$ which is defined on $\mathcal{T}^c_{h_c}$. Since $\lambda_{2n} \in \mathcal{Z}$, we can let $\lambda^*_{2n} \in V_{ch}/P_0$ be the discrete harmonic extension of $\lambda_{2n}|_{\Gamma_c}$ in Ω_c. Then we know that $\nabla\lambda^*_{2n} \in \mathbf{V}_{1h}$ and

$$\nabla\lambda^*_{2n} \times \mathbf{n}_c|_{\partial\Omega_c} = \mathbf{R}^H_n\lambda_{2n} \times \mathbf{n}_c|_{\partial\Omega_c}.$$

According to (5.1), we have

$$\int_{\Omega_c} \mathbf{R}^H_n\lambda_{1n} \cdot (\mathbf{R}^H_n\lambda_{2n} - \nabla\lambda^*_{2n}) + \tfrac{\tau_n}{\mu\sigma}(\nabla \times \mathbf{R}^H_n\lambda_{1n}) \cdot \nabla \times (\mathbf{R}^H_n\lambda_{2n} - \nabla\lambda^*_{2n})dV = 0,$$

then we can obtain that

$$\begin{aligned}&\int_{\Omega_c} \mathbf{R}^H_n\lambda_{1n} \cdot \mathbf{R}^H_n\lambda_{2n} + \tfrac{\tau_n}{\mu\sigma}(\nabla \times \mathbf{R}^H_n\lambda_{1n}) \cdot (\nabla \times \mathbf{R}^H_n\lambda_{2n})dV\\ &= \int_{\Omega_c} \mathbf{R}^H_n\lambda_{1n} \cdot \nabla\lambda^*_{2n}dV = \int_{\Gamma_c} \mathbf{R}^H_n\lambda_{1n} \cdot \mathbf{n}_c\lambda_{2n}dA.\end{aligned}$$

So

$$\langle S^h_{1n}\lambda_{1n}, \lambda_{2n}\rangle = \int_{\Omega_c} \left(\mathbf{R}^H_n\lambda_{1n} \cdot \mathbf{R}^H_n\lambda_{2n} + \tfrac{\tau_n}{\mu\sigma}(\nabla \times \mathbf{R}^H_n\lambda_{1n}) \cdot (\nabla \times \mathbf{R}^H_n\lambda_{2n})\right) dV.$$

From the previous formulation, we can see that

$$\langle S^h_{1n}\lambda_{1n}, \lambda_{1n}\rangle = \int_{\Omega_c} \left(\mathbf{R}^H_n\lambda_{1n} \cdot \mathbf{R}^H_n\lambda_{1n} + \tfrac{\tau_n}{\mu\sigma}(\nabla \times \mathbf{R}^H_n\lambda_{1n}) \cdot (\nabla \times \mathbf{R}^H_n\lambda_{1n})\right) dV \geq 0,\ \forall\ \lambda_{1n} \in \mathcal{Z},$$

and $\langle S^h_{1n}\lambda_{1n}, \lambda_{2n}\rangle = \langle S^h_{1n}\lambda_{2n}, \lambda_{1n}\rangle, \quad \forall\ \lambda_{1n},\ \lambda_{2n} \in \mathcal{Z}.$

So the operators $S^h_{1n} : \mathcal{Z} \to \mathcal{Z}'$ is symmetric and positive semi-definite for $n = 1, \ldots, M$. $\square$

Lemma 5.3. *The operator $S_{2n}^h : \mathcal{Z} \to \mathcal{Z}'$ is symmetric and positive definite for $n = 1, \ldots, M$.*

Proof: For any $\lambda_{1n}, \lambda_{2n} \in \mathcal{Z}$, since

$$\langle S_{2n}^h \lambda_{1n}, \lambda_{2n} \rangle = \int_{\Omega_{e_1}} \nabla R_n^{\psi_{e_1}} \lambda_{1n} \cdot \nabla R_n^{\psi_{e_1}} \lambda_{2n} dV, \quad \text{for } n = 1, \ldots, M,$$

it is easy to see that the operators S_{2n}^h are symmetric. Moreover, according to Poincaré inequality, we know that

$$\begin{aligned} \|R_n^{\psi_{e_1}} \lambda_{1n}\|^2_{H^1(\Omega_{e_1})/P_0} = & \inf_{w \in P_0} \|R_n^{\psi_{e_1}} \lambda_{1n} - w\|^2_{H^1(\Omega_{e_1})} \\ & \leq C \inf_{w \in P_0} \left(|R_n^{\psi_{e_1}} \lambda_{1n} - w|_{H^1(\Omega_{e_1})} + | \textstyle\int_{\Omega_{e_1}} (R_n^{\psi_{e_1}} \lambda_{1n} - w) dV| \right)^2, \end{aligned}$$

where C is a positive constant. Taking $w = \frac{\int_{\Omega_{e_1}} R_n^{\psi_{e_1}} \lambda_{1n} dV}{\int_{\Omega_{e_1}} 1 dV}$, it follows that

$$\|R_n^{\psi_{e_1}} \lambda_{1n}\|^2_{H^1(\Omega_{e_1})/P_0} \leq C |R_n^{\psi_{e_1}} \lambda_{1n}|^2_{H^1(\Omega_{e_1})}.$$

By using the trace theorem, we get

$$\begin{aligned} \langle S_{2n}^h \lambda_{1n}, \lambda_{1n} \rangle & = \textstyle\int_{\Omega_{e_1}} \nabla R_n^{\psi_{e_1}} \lambda_{1n} \cdot \nabla R_n^{\psi_{e_1}} \lambda_{1n} dV \\ & = |R_n^{\psi_{e_1}} \lambda_{1n}|^2_{H^1(\Omega_{e_1})} \geq \frac{1}{C} \|R_n^{\psi_{e_1}} \lambda_{1n}\|^2_{H^1(\Omega_{e_1})/P_0}, \\ & \geq C_n \|\lambda_{1n}\|^2_{H^{\frac{1}{2}}(\partial\Omega_{e_1})/P_0}, \end{aligned}$$

where C_n is the positive constant depending on n. So the operator $S_{2n}^h : \mathcal{Z} \to \mathcal{Z}'$ is symmetric and positive definite for $n = 1, \ldots, M$. □

Lemma 5.4. *The operator $S_{3n}^h : \mathcal{Z} \to \mathcal{Z}'$ is symmetric positive semi-definite for $n = 1, \ldots, M$.*

Proof: For any $\lambda_{1n}, \lambda_{2n} \in \mathcal{Z}$,

$$\begin{aligned} \langle S_{3n}^h \lambda_{1n}, \lambda_{2n} \rangle & = - \textstyle\int_{\Gamma_e} \frac{\partial \lambda_{1n}^e}{\partial \mathbf{n}_e} \lambda_{2n}^0 dA \\ & = \sum_{l=1}^{\infty} \frac{(l+1)}{R} \left(\sum_{k=1}^{l} (q_{1n,1l}^k q_{2n,1l}^k + q_{1n,2l}^k q_{2n,2l}^k) + q_{1n,0l}^0 q_{2n,0l}^0 \right), \end{aligned}$$

so we have

$$\langle S_{3n}^h \lambda_{1n}, \lambda_{1n} \rangle = \sum_{l=1}^{\infty} \frac{(l+1)}{R} (\sum_{k=1}^{l} (|q_{1n,1l}^k|^2 + |q_{1n,2l}^k|^2) + |q_{1n,0l}^0|^2) \geq 0$$

and

$$\langle S_{3n}^h \lambda_{1n}, \lambda_{2n} \rangle = \langle S_{3n}^h \lambda_{2n}, \lambda_{1n} \rangle.$$

Therefore, the operator $S_{3n}^h : \mathcal{Z} \to \mathcal{Z}'$ is symmetric positive semi-definite for $n = 1, \ldots, M$. □

Lemma 5.5. *For any $\lambda_{1n} \in \mathcal{Z}$, we have*

$$\langle S_{3n}^h \lambda_{1n}, \lambda_{1n} \rangle \leq C_1^* \langle S_{2n}^h \lambda_{1n}, \lambda_{1n} \rangle, \quad \text{for } n = 1, \ldots, M,$$

where the positive constant C_1^* is independent of n, h_c and h_e.

Proof: For any $\lambda_{1n} \in \mathcal{Z}$,

$$\langle S_{3n}^h \lambda_{1n}, \lambda_{1n} \rangle = \sum_{l=1}^{\infty} \frac{(l+1)}{R} \left(\sum_{k=1}^{l} (|q_{1n,1l}^k|^2 + |q_{1n,2l}^k|^2) + |q_{1n,0l}^0|^2 \right).$$

Since

$$\begin{aligned}
\|\lambda_{1n}|_{\Gamma_e}\|_{H^{\frac{1}{2}}(\Gamma_e)/P_0} &= \inf_{w \in P_0} \|\lambda_{1n}^0|_{\Gamma_e} - w\|_{H^{\frac{1}{2}}(\Gamma_e)} \\
&= \inf_{w \in P_0} R \left(\sum_{l=1}^{\infty} (l+1) \left(\sum_{k=1}^{l} (|q_{1n,1l}^k|^2 + |q_{1n,2l}^k|^2) + |q_{1n,0l}^0|^2 \right) + 4\pi (w)^2 \right)^{\frac{1}{2}} \\
&= R \left(\sum_{l=1}^{\infty} (l+1) \left(\sum_{k=1}^{l} (|q_{1n,1l}^k|^2 + |q_{1n,2l}^k|^2) + |q_{1n,0l}^0|^2 \right) \right)^{\frac{1}{2}},
\end{aligned}$$

we have

$$\langle S_{3n}^h \lambda_{1n}, \lambda_{1n} \rangle = \frac{1}{R^3} \|\lambda_{1n}|_{\Gamma_e}\|^2_{H^{\frac{1}{2}}(\Gamma_e)/P_0} \leq \frac{1}{R^3} \|\lambda_{1n}\|^2_{H^{\frac{1}{2}}(\partial\Omega_{e_1})/P_0} \leq C_{1n}^* \langle S_{2n}^h \lambda_{1n}, \lambda_{1n} \rangle,$$

where C_{1n}^* is independent of h_c and h_e.

Let $C_1^* = \max_{1 \leq n \leq M} C_{1n}^*$. This completes the proof. $\square$

Lemma 5.6. *For any $\lambda_{1n} \in \mathcal{Z}$, we have*

$$\langle S_{1n}^h \lambda_{1n}, \lambda_{1n} \rangle \leq C_2^* \langle S_{2n}^h \lambda_{1n}, \lambda_{1n} \rangle, \quad \text{for } n = 1, \ldots, M,$$

where the positive constant C_2^* is independent of n, h_c and h_e.

Proof: According to [4], we know that

$$\begin{aligned}
\langle S_{1n}^h \lambda_{1n}, \lambda_{1n} \rangle &\leq C \|\nabla \lambda_{1n} \times \mathbf{n}_c\|^2_{\mathbf{H}^{-\frac{1}{2}}(div_{\Gamma_c}, \Gamma_c)} \\
&= C \left(\|\nabla \lambda_{1n} \times \mathbf{n}_c\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_c)} + \|div_{\Gamma_c}(\nabla \lambda_{1n} \times \mathbf{n}_c)\|^2_{H^{-\frac{1}{2}}(\Gamma_c)} \right) \\
&= C \|\nabla R_n^{\psi_{e_1}} \lambda_{1n} \times \mathbf{n}_c\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_c)}, \quad \forall\, \lambda_{1n} \in \mathcal{Z},
\end{aligned}$$

where C is independent of h_c.

Let us introduce the Hilbert space $\mathcal{X}_{00,\Gamma_c}$ defined in [3],

$$\mathcal{X}_{00,\Gamma_c} := \{\varsigma \in \mathbf{H}_{00}^{-\frac{1}{2}}(\Gamma_c) \mid \varsigma \cdot \mathbf{n}|_{\Gamma_c} = 0 \text{ and } div_{\Gamma_c} \varsigma \in H_{00}^{-\frac{1}{2}}(\Gamma_c)\}.$$

With the result given in [3], we know that $(\nabla R_n^{\psi_{e_1}} \lambda_{1n} \times \mathbf{n}_c)|_{\Gamma_c} \in \mathcal{X}_{00,\Gamma_c}$ and

$$\|(\nabla R_n^{\psi_{e_1}} \lambda_{1n} \times \mathbf{n}_c)|_{\Gamma_c}\|_{\mathcal{X}_{00,\Gamma_c}} \leq C^* \|\nabla R_n^{\psi_{e_1}} \lambda_{1n}\|_{\mathbf{H}(\mathbf{curl}, \Omega_{e_1})}.$$

Since

$$\|(\nabla R_n^{\psi_{e_1}} \lambda_{1n} \times \mathbf{n}_c)|_{\Gamma_c}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_c)} \leq \overline{C} \|(\nabla R_n^{\psi_{e_1}} \lambda_{1n} \times \mathbf{n}_c)|_{\Gamma_c}\|_{\mathbf{H}_{00}^{-\frac{1}{2}}(\Gamma_c)},$$

where $\overline{C}$ depends on the area of Γ_e and the distance between Γ_c and Γ_e, we have

$$\|(\nabla R_n^{\psi_{e_1}} \lambda_{1n} \times \mathbf{n}_c)|_{\Gamma_c}\|_{H^{-\frac{1}{2}}(\Gamma_c)} \leq C \|\nabla R_n^{\psi_{e_1}} \lambda_{1n}\|_{\mathbf{H}(\mathbf{curl}, \Omega_{e_1})} = C \|\nabla R_n^{\psi_{e_1}} \lambda_{1n}\|_{\mathbf{L}^2(\Omega_{e_1})}.$$

Then we obtain that

$$\langle S_{1n}^h\lambda_{1n},\lambda_{1n}\rangle \le C\|\nabla R_n^{\psi_{e_1}}\lambda_{1n}\times\mathbf{n}_c\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_c)} \le C_{2n}^*\|\nabla R_n^{\psi_{e_1}}\lambda_{1n}\|^2_{\mathbf{L}^2(\Omega_{e_1})} = C_{2n}^*\langle S_{2n}^h\lambda_{1n},\lambda_{1n}\rangle,$$

where C_{2n}^* is independent of h_c and h_e.

Let $C_2^* = \max\limits_{1\le n\le M} C_{2n}^*$. This completes the proof. □

Now we can give the convergence of the preconditioned Richardson iterative method (5.12) at each time step. The proof of this theorem depends on the previous Lemmata.

Theorem 5.1. *If $0 < \min\limits_m \vartheta_{n,m} \le \max\limits_m \vartheta_{n,m} < 2/(1+C_1^*+C_2^*)$ for $n=1,2,\ldots,M$, then the preconditioned Richardson iterative method at each time step converges with a rate independent of h_c and h_e.*

Proof: The discrete Steklov-Poincaré operators S_{1n}^h, S_{2n}^h and S_{3n}^h are linear operators from the finite dimensional space $\mathcal{Z}$ into its dual $\mathcal{Z}'$. Let $\{\mathbf{x}_s\}$, $s=1,\ldots,M_h$, be a basis of $\mathcal{Z}$, then we define the matrices associated with the finite dimensional operators S_{1n}^h, S_{2n}^h and S_{3n}^h:

$$(A_{in}^h\widehat{\lambda}_{1n},\widehat{\lambda}_{2n}) := \langle S_{in}^h\lambda_{1n},\lambda_{2n}\rangle, \quad \forall\ \widehat{\lambda}_{1n},\widehat{\lambda}_{2n}\in\mathbb{R}^{M_h}, i=1,2,3, \tag{5.13}$$

where $(\cdot,\cdot)$ denotes the Euclidean scalar product in $\mathbb{R}^{M_h}$ and

$$\lambda_{1n} = \sum_{k=1}^{M_h}\widehat{\lambda}_{1nk}\mathbf{x}_k, \quad \lambda_{2n} = \sum_{k=1}^{M_h}\widehat{\lambda}_{2nk}\mathbf{x}_k.$$

According to Lemma 5.2-5.4, we can get that the operators S_n^h are symmetric and positive definite. By using Lemma 5.5 and 5.6, we have

$$\tfrac{1}{1+C_1^*+C_2^*}\langle S_n^h\lambda_{1n},\lambda_{1n}\rangle \le \langle S_{2n}^h\lambda_{1n},\lambda_{1n}\rangle \le \langle S_n^h\lambda_{1n},\lambda_{1n}\rangle, \qquad \forall\ \lambda_{1n}\in\mathcal{Z},\ \text{for } n=1,\ldots,M.$$

Since

$$\begin{aligned}\lambda_{n,m+1}-\lambda_n^\dagger &= (1-\vartheta_{n,m}(S_{2n}^h)^{-1}S_n^h)(\lambda_{n,m}-\lambda_n^\dagger)\\ &= \prod_{k=0}^{m}(1-\vartheta_{n,k}(S_{2n}^h)^{-1}S_n^h)(\lambda_{n,0}-\lambda_n^\dagger),\end{aligned}$$

we can obtain the matrix form:

$$\begin{aligned}\widehat{\lambda}_{n,m+1}-\widehat{\lambda}_n^\dagger &= (I-\vartheta_{n,m}(A_{2n}^h)^{-1}A_n^h)(\widehat{\lambda}_{n,m}-\widehat{\lambda}_n^\dagger)\\ &= \prod_{k=0}^{m}(I-\vartheta_{n,k}(A_{2n}^h)^{-1}A_n^h)(\widehat{\lambda}_{n,0}-\widehat{\lambda}_n^\dagger).\end{aligned}$$

Then

$$\|\widehat{\lambda}_{n,m+1}-\widehat{\lambda}_n^\dagger\| \le \nu_n^{m+1}\|\widehat{\lambda}_{n,0}-\widehat{\lambda}_n^\dagger\|,$$

where $\|\cdot\|$ is the norm of vector and

$$\nu_n = \max(1-\min_m\vartheta_{n,m}, (1+C_1^*+C_2^*)\max_m\vartheta_{n,m}-1) < 1,$$

when $0<\min\limits_m\vartheta_{n,m}\le\max\limits_m\vartheta_{n,m}<2/(1+C_1^*+C_2^*)$.

It follows that

$$\lim_{m\to\infty}\|\widehat{\lambda}_{n,m+1}-\widehat{\lambda}_n^\dagger\|=0,\quad \text{for } n=1,2,\ldots,M.$$

Therefore,

$$\lim_{m\to\infty}\|\lambda_{n,m+1}-\lambda_n^\dagger\|_{H^{\frac{1}{2}}(\partial\Omega_{e_1})/P_0}=0,\quad \text{for } n=1,2,\ldots,M,$$

and the convergence rate of the preconditioned Richardson iterative method at each time step is independent of h_c and h_e. $\square$

5.3 Error estimate for the fully discrete scheme

Let $\overline{\Omega}_b=\overline{\Omega}_c\cup\overline{\Omega}_{e_1}$ and

$$\begin{aligned}\mathbf{X}:= \quad &\{\mathbf{v}:\mathbf{v}=\mathbf{w}\text{ in }\Omega_c\text{ for some }\mathbf{w}\in\mathbf{V}_1\text{ and }\mathbf{v}=\nabla\varphi\text{ in }\Omega_{e_1}\\ &\text{for some }\varphi\in V_2\text{ such that }\mathbf{w}\times\mathbf{n}_c=\nabla\varphi\times\mathbf{n}_c\text{ on }\Gamma_c\}.\end{aligned}$$

We denote the finite element space of $\mathbf{X}$ by

$$\begin{aligned}\mathbf{X}_h:= \quad &\{\mathbf{v}_h:\mathbf{v}_h=\mathbf{w}_h\text{ in }\Omega_c\text{ for some }\mathbf{w}_h\in\mathbf{V}_{1h}\text{ and }\mathbf{v}_h=\nabla\varphi_h\text{ in }\Omega_{e_1}\\ &\text{for some }\varphi_h\in V_{2h}\text{ such that }\mathbf{w}_h\times\mathbf{n}_c=\nabla\varphi_h\times\mathbf{n}_c\text{ on }\Gamma_c\}.\end{aligned}$$

Let $\mathbf{H}_0^r=\begin{cases}\mathbf{H}_0^r,\ \text{in }\Omega_c\\ \nabla\psi_2,\ \text{in }\Omega_e\end{cases}$ be the solution of the problem (2.18). Consider the following variational formulation, which is the problem (2.18) restricted in Ω_b.

Find $\mathbf{H}_0\in\mathbf{L}^2((0,T),\mathbf{X})$ such that

$$\begin{cases}\int_{\Omega_b}\frac{\partial\mathbf{H}_0}{\partial t}\cdot\mathbf{w}dV+\frac{1}{\mu\sigma}\int_{\Omega_c}(\nabla\times\mathbf{H}_0)\cdot(\nabla\times\mathbf{w})dV-\int_{\Gamma_e}\frac{\partial\mathcal{G}_e(\psi_2|_{\Gamma_e})}{\partial t}\varphi dA\\ =-\int_{\Omega_c}\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{w}dV+\int_{\Gamma_c}(\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{n}_c)\varphi dA,\qquad\forall\,\mathbf{w}\in\mathbf{X},\\ \mathbf{H}_0(\mathbf{x},0)=\mathbf{0},\quad\mathbf{x}\in\Omega_b,\end{cases}\tag{5.14}$$

where $\mathbf{w}=\begin{cases}\mathbf{w},\ \text{in }\Omega_c\\ \nabla\varphi,\ \text{in }\Omega_{e_1}\end{cases}$ and $\mathbf{H}_0=\begin{cases}\mathbf{H}_0,\ \text{in }\Omega_c\\ \nabla\psi_1,\ \text{in }\Omega_{e_1}\end{cases}$. The solution of the problem (5.14) is equal to $\mathbf{H}_0^r|_{\Omega_b}$. Moreover, it is easy to see that

$$-\int_{\Gamma_e}\mu\frac{\partial\mathcal{G}_e(\psi_2|_{\Gamma_e})}{\partial t}dA+\int_{\Gamma_c}(\mu\frac{\partial\mathbf{H}_s}{\partial t}\cdot\mathbf{n}_c)dA=0.$$

Let $(\mathbf{H}_{0n}^r,\psi_{1n}^r,\psi_{2n})$ be the solution of the problem (3.11). The variational form of the problem (3.1) restricted in Ω_b is as follows:

Find $\mathbf{H}_{0n}\in\mathbf{X}$, for $n=1,2,\ldots,M$, such that

$$\begin{cases}\int_{\Omega_b}\frac{\mathbf{H}_{0n}-\mathbf{H}_{0n-1}}{\tau_n}\cdot\mathbf{w}dV+\frac{1}{\mu\sigma}\int_{\Omega_c}(\nabla\times\mathbf{H}_{0n})\cdot(\nabla\times\mathbf{w})dV-\int_{\Gamma_e}\left(\frac{\mathcal{G}_e(\psi_{2n}|_{\Gamma_e})-\mathcal{G}_e(\psi_{2n-1}|_{\Gamma_e})}{\tau_n}\right)\varphi dA\\ =-\int_{\Omega_c}\frac{\partial\mathbf{H}_s}{\partial t}(t_n)\cdot\mathbf{w}dV+\int_{\Gamma_c}(\frac{\partial\mathbf{H}_s}{\partial t}(t_n)\cdot\mathbf{n}_c)\varphi dA,\quad\forall\,\mathbf{w}\in\mathbf{X},\\ \mathbf{H}_{00}=\mathbf{0},\quad\mathbf{x}\in\Omega_b.\end{cases}\tag{5.15}$$

The solution of the problem (5.15) is equal to $\left\{\begin{array}{l}\mathbf{H}_{0n}^r, \text{ in } \Omega_c \\ \nabla\psi_{1n}^r, \text{ in } \Omega_{e_1}\end{array}\right.$, for $n = 1, 2, \ldots, M$. By using the system (3.1), we can verify that

$$\int_{\Gamma_e} \mu \left(\frac{\mathcal{G}_e(\psi_{2n}|_{\Gamma_e}) - \mathcal{G}_e(\psi_{2n-1}|_{\Gamma_e})}{\tau_n} \right) dA + \int_{\Gamma_c} (\mu \frac{\partial \mathbf{H}_s}{\partial t}(t_n) \cdot \mathbf{n}_c) dA = 0, \text{ for } n = 1, 2, \ldots, M.$$

Let the solution of the discrete problem of (3.11) be $(\mathbf{H}_{0n}^{r,h}, \psi_{1n}^{r,h}, \psi_{2n}^h)$, then we can introduce the discrete formulation of the problem (5.15).

Find $\mathbf{H}_{0n}^h \in \mathbf{X}_h$, for $n = 1, 2, \ldots, M$, such that

$$\left\{\begin{array}{l} \int_{\Omega_b} \frac{\mathbf{H}_{0n}^h - \mathbf{H}_{0n-1}^h}{\tau_n} \cdot \mathbf{w}_h dV + \frac{1}{\mu\sigma} \int_{\Omega_c} (\nabla \times \mathbf{H}_{0n}^h) \cdot (\nabla \times \mathbf{w}_h) dV - \int_{\Gamma_e} \left(\frac{\mathcal{G}_e(\psi_{2n}^h|_{\Gamma_e}) - \mathcal{G}_e(\psi_{2n-1}^h|_{\Gamma_e})}{\tau_n} \right) \varphi_h dA \\ = -\int_{\Omega_c} \frac{\partial \mathbf{H}_s}{\partial t}(t_n) \cdot \mathbf{w}_h dV + \int_{\Gamma_c} (\frac{\partial \mathbf{H}_s}{\partial t}(t_n) \cdot \mathbf{n}_c) \varphi_h dA, \quad \forall\, \mathbf{w}_h \in \mathbf{X}_h, \\ \mathbf{H}_{00}^h = \mathbf{0}, \quad \mathbf{x} \in \Omega_b, \end{array}\right. \tag{5.16}$$

where $\mathbf{w}_h = \left\{\begin{array}{l}\mathbf{w}_h, \text{ in } \Omega_c \\ \nabla\varphi_h, \text{ in } \Omega_{e_1}\end{array}\right.$ and the solution of the problem (5.16) is $\mathbf{H}_{0n}^h = \left\{\begin{array}{l}\mathbf{H}_{0n}^{r,h}, \text{ in } \Omega_c \\ \nabla\psi_{1n}^{r,h}, \text{ in } \Omega_{e_1}\end{array}\right.$ for $n = 0, 1, \ldots, M$.

Since

$$\int_{\Gamma_e} \frac{\partial \psi_{2n}^h}{\partial \mathbf{n}_e} dA - \int_{\Gamma_c} (\mathbf{H}_{0n} \cdot \mathbf{n}_c) dA = 0,$$

we can get that

$$\int_{\Gamma_c} (\mu \frac{\partial \mathbf{H}_s}{\partial t}(t_n) \cdot \mathbf{n}_c) dA + \int_{\Gamma_e} \mu \left(\frac{\mathcal{G}_e(\psi_{2n}^h|_{\Gamma_e}) - \mathcal{G}_e(\psi_{2n-1}^h|_{\Gamma_e})}{\tau_n} \right) dA = 0, \text{ for } n = 1, 2, \ldots, M.$$

In the domain Ω_b, the backward Euler-DtN alternating algorithm proposed in subsection 4.3 is actually used to solve the following problem:

Find $\mathbf{H}_{0n}^{\dagger h} \in \mathbf{X}_h$, for $n = 1, 2, \ldots, M$, such that

$$\left\{\begin{array}{l} \int_{\Omega_c} \frac{\mathbf{H}_{0n}^{\dagger h} - \widetilde{\mathbf{H}}_{0n-1}^h}{\tau_n} \cdot \mathbf{w}_h dV + \frac{1}{\sigma\mu} \int_{\Omega_c} (\nabla \times \mathbf{H}_{0n}^{\dagger h}) \cdot (\nabla \times \mathbf{w}_h) dV + \int_{\Omega_{e_1}} \frac{\nabla\psi_{1n}^{\dagger h} - \nabla\widetilde{\psi}_{1n-1}^h}{\tau_n} \cdot \nabla\varphi_h dV \\ - \int_{\Gamma_e} \frac{\mathcal{G}_e(\psi_{2n}^{\dagger h}|_{\Gamma_e}) - \mathcal{G}_e(\widetilde{\psi}_{2n-1}^h|_{\Gamma_e})}{\tau_n} \varphi_h dA = -\int_{\Omega_c} \frac{\partial \mathbf{H}_s}{\partial t}(t_n) \cdot \mathbf{w}_h dV + \int_{\Gamma_c} \frac{\partial \mathbf{H}_s}{\partial t}(t_n) \cdot \mathbf{n}_c \varphi_h dA, \ \forall\, \mathbf{w}_h \in \mathbf{X}_h, \\ \mathbf{H}_{00}^{\dagger h} = \mathbf{0}, \quad \mathbf{x} \in \Omega_b, \end{array}\right. \tag{5.17}$$

where $\mathbf{H}_{0n}^{\dagger h} = \left\{\begin{array}{l}\mathbf{H}_{0n}^{\dagger h}, \text{ in } \Omega_c \\ \nabla\psi_{1n}^{\dagger h}, \text{ in } \Omega_{e_1}\end{array}\right.$ and $\widetilde{\mathbf{H}}_{0n}^h$ has been defined in subsection 4.3 for $n = 0, 1, 2, \ldots, M$. It can be verified that

$$\int_{\Gamma_e} \frac{\mathcal{G}_e(\psi_{2n}^{\dagger h}|_{\Gamma_e}) - \mathcal{G}_e(\widetilde{\psi}_{2n-1}^h|_{\Gamma_e})}{\tau_n} dA + \int_{\Gamma_c} \frac{\partial \mathbf{H}_s}{\partial t}(t_n) \cdot \mathbf{n}_c dA = 0, \text{ for } n = 1, 2, \ldots, M.$$

Let $\mathbf{H}^a(\mathbf{curl}, \Omega_b) = \{\mathbf{v} \in \mathbf{H}^a(\Omega_b) \mid \nabla \times \mathbf{v} \in \mathbf{H}^a(\Omega_b)\}$ $(a > 0)$ equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{H}^a(\mathbf{curl}, \Omega_b)} = \left(\|\mathbf{v}\|_{\mathbf{H}^a(\Omega_b)}^2 + \|\nabla \times \mathbf{v}\|_{\mathbf{H}^a(\Omega_b)}^2 \right)^{\frac{1}{2}},$$

then we give the error estimate between $\mathbf{H}_{0n}^h$ and $\mathbf{H}_0(t_n)$.

Theorem 5.2. *Assume that the solution $\mathbf{H}_0$ of the problem (5.14) is regular enough. Let $\{\mathbf{H}_{0n}^h\}$ be the approximation solution defined by (5.16). $\psi_2(t_n)$ and ψ_{2n}^h have been used in (5.14) and (5.16). $h = \max\{h_c, h_e\}$ and $\tau = \max\limits_n \tau_n$. Then we have*

$$\begin{aligned} & \|\mathbf{H}_{0n}^h - \mathbf{H}_0(t_n)\|_{\mathbf{H}(\mathbf{curl},\Omega_c)} + \|\nabla\psi_{1n}^h - \nabla\psi_1(t_n)\|_{\mathbf{L}^2(\Omega_{e_1})} + \|\psi_{2n}^h - \psi_2(t_n)\|_{H^{\frac{1}{2}}(\Gamma_e)} \\ \leq \;\; & C\Big(h^s(1+t_n)\int_0^{t_n}\|\mathbf{H}_{0t}(l)\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)}dl + h^{r-1}\int_0^{t_n}\|\psi_{1t}(l)\|_{H^r(\Omega_{e_1})/P_0}dl \\ & +\tau\int_0^{t_n}\|\psi_{2tt}(l)\|_{H^{\frac{1}{2}}(\Gamma_e)}dl + \tau\int_0^{t_n}\|\mathbf{H}_{0tt}(l)\|_{\mathbf{L}^2(\Omega_b)}dl\Big), \qquad \text{for } n = 1,2,\ldots,M, \end{aligned}$$

where $\frac{1}{2} < s \leq 1$, $1 \leq r \leq r_0$ for some $r_0 \geq 2$.

Proof: We divide the proof into three steps.

Step 1. Define a mapping $\mathbf{P}_1 : \mathbf{X} \to \mathbf{X}_h$

Define

$$A(\mathbf{D},\mathbf{B}) = \int_{\Omega_c} \nabla\times\mathbf{D}\cdot\nabla\times\mathbf{B}dV \qquad \forall\, \mathbf{D},\mathbf{B} \in \mathbf{H}(\text{curl},\Omega_c),$$

and

$$b(\mathbf{B},q) = -\int_{\Omega_c} \mathbf{B}\cdot\nabla q dV \qquad \forall\, \mathbf{B} \in \mathbf{H}(\text{curl},\Omega_c),\ q \in H_0^1(\Omega_c).$$

Let U_h be the finite element space of $H_0^1(\Omega_c)$ defined on $\mathcal{T}_{h_c}^c$. Then we define a mapping $\mathbf{P}_c : \mathbf{V}_1 \to \mathbf{V}_{1h}$. For any $\mathbf{v} \in \mathbf{V}_1$, $\mathbf{P}_c(\mathbf{v})$ satisfies the following system.

Find $(\mathbf{P}_c(\mathbf{v}), p^h) \in \mathbf{V}_{1h} \times U_h$ such that

$$\begin{cases} A(\mathbf{P}_c(\mathbf{v}),\mathbf{B}_h) + b(\mathbf{B}_h,p^h) = A(\mathbf{v},\mathbf{B}_h) + b(\mathbf{B}_h,p), & \forall\, \mathbf{B}_h \in \mathbf{V}_{1h}^0, \\ b(\mathbf{P}_c(\mathbf{v}),q^h) = b(\mathbf{v},q^h), & \forall\, q^h \in U_h, \\ \mathbf{P}_c(\mathbf{v})\times\mathbf{n}_c = \mathbf{v}\times\mathbf{n}_c, \quad p^h = 0, & \text{on } \Gamma_c, \end{cases}$$

where $p \in H_0^1(\Omega_c)$. According to [12], we have

$$\|\mathbf{P}_c(\mathbf{v}) - \mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_c)} + \|p - p^h\|_{H_0^1(\Omega_c)} \leq Ch^s(\|\mathbf{v}\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)} + \|p\|_{H^{1+s}(\Omega_c)}),$$

for $\mathbf{v} \in \mathbf{H}^s(\mathbf{curl},\Omega_c)$, $p \in H^{1+s}(\Omega_c)$, for some $1/2 < s \leq 1$.

Next we can define a mapping $\widehat{P}_e : \widetilde{V}_2 \to \widetilde{V}_{2h}$. For any $u \in \widetilde{V}_2$, $\widehat{P}_e(u)$ satisfies the following system:

Find $\widehat{P}_e(u) \in \widetilde{V}_{2h}$ such that

$$\begin{cases} \int_{\Omega_{e_1}} \nabla\widehat{P}_e(u)\cdot\nabla\widehat{\varphi}_h dV = \int_{\Omega_{e_1}} \nabla u\cdot\nabla\widehat{\varphi}_h dV, & \forall\, \widehat{\varphi}_h \in \widetilde{V}_{2h}^0, \\ \widehat{P}_e(u) = u, & \text{on } \partial\Omega_{e_1}. \end{cases}$$

According to [19], we know

$$\|\nabla\widehat{P}_e(u) - \nabla u\|_{L^2(\Omega_{e_1})} \leq Ch^{r-1}\|u\|_{H^r(\Omega_{e_1})}.$$

Define the mapping $\mathbf{P}_e : V_2 \to V_{2h}$. For any $\widehat{u} \in V_2$,

$$\mathbf{P}_e(\widehat{u}) = \left[\widehat{P}_e(u)\right],$$

where $u \in \widetilde{V}_2$ and $\widehat{u} = [u]$. Then we can get

$$\|\nabla \mathbf{P}_e(\widehat{u}) - \nabla \widehat{u}\|_{L^2(\Omega_{e_1})} \le Ch^{r-1}\|\widehat{u}\|_{H^r(\Omega_{e_1})/P_0}.$$

So we define the mapping $\mathbf{P}_1 : \mathbf{X} \to \mathbf{X}_h$ such that

$$\mathbf{P}_1(\mathbf{v}) = \begin{cases} \mathbf{P}_c(\mathbf{v}), & \text{in } \Omega_c \\ \nabla \mathbf{P}_e(\varphi), & \text{in } \Omega_{e_1} \end{cases}, \quad \forall\ \mathbf{v} \in \mathbf{X}.$$

Step 2. Give the error estimate between $\mathbf{H}_0(t_n)$ and $\mathbf{P}_1(\mathbf{H}_0(t_n))$

Since $\mathbf{H}_0(t_n) \in \mathbf{X}$, we have

$$\begin{aligned} &\|\mathbf{P}_c(\mathbf{H}_0(t_n)|_{\Omega_c}) - \mathbf{H}_0(t_n)|_{\Omega_c}\|_{\mathbf{H}(\mathbf{curl},\Omega_c)} + \|p - p^h\|_{H_0^1(\Omega_c)} \\ &\le Ch^s(\|\mathbf{H}_0(t_n)|_{\Omega_c}\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)} + \|p\|_{H^{1+s}(\Omega_c)}) \end{aligned} \tag{5.18}$$

and

$$\|\nabla \mathbf{P}_e(\psi_1(t_n)) - \nabla \psi_1(t_n)\|_{L^2(\Omega_{e_1})} \le Ch^{r-1}\|\psi_1(t_n)\|_{H^r(\Omega_{e_1})/P_0}, \tag{5.19}$$

where $\mathbf{H}_0(t_n)|_{\Omega_{e_1}} = \nabla \psi_1(t_n)$.

Moreover, since $\nabla \cdot \mathbf{H}_0(t_n) = 0$, according to [12], we can get that the p of (5.18) is equal to zero. Then

$$\|\mathbf{P}_c(\mathbf{H}_0(t_n)|_{\Omega_c}) - \mathbf{H}_0(t_n)|_{\Omega_c}\|_{\mathbf{H}(\mathbf{curl},\Omega_c)} \le Ch^s\|\mathbf{H}_0(t_n)|_{\Omega_c}\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)}. \tag{5.20}$$

The error estimate between $\mathbf{H}_0(t_n)$ and $\mathbf{P}_1(\mathbf{H}_0(t_n))$ can be obtained.

$$\begin{aligned} &\|\mathbf{P}_1(\mathbf{H}_0(t_n)) - \mathbf{H}_0(t_n)\|_{\mathbf{H}(\mathbf{curl},\Omega_b)} \\ &\le C\left(h^s\|\mathbf{H}_0(t_n)\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)} + h^{r-1}\|\psi_1(t_n)\|_{H^r(\Omega_{e_1})/P_0}\right) \\ &\le C\left(h^s\int_0^{t_n}\|\mathbf{H}_{0t}(l)\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)}dl + h^{r-1}\int_0^{t_n}\|\psi_{1t}(l)\|_{H^r(\Omega_{e_1})/P_0}dl\right). \end{aligned} \tag{5.21}$$

Step 3. Give the error estimate between $\mathbf{H}_0(t_n)$ and $\mathbf{H}_{0n}^h$

Let

$$\mathbf{H}_{0n}^h - \mathbf{H}_0(t_n) = \mathbf{H}_{0n}^h - \mathbf{P}_1(\mathbf{H}_0(t_n)) + \mathbf{P}_1(\mathbf{H}_0(t_n)) - \mathbf{H}_0(t_n) = \varpi^n + \varrho^n.$$

We denote $\frac{(\varpi^n - \varpi^{n-1})}{\tau_n}$ by $\overline{\partial}_t\varpi^n$. Then it is easy to see that ϖ^n satisfies the following equation

$$\begin{aligned} &\int_{\Omega_b}\overline{\partial}_t\varpi^n\cdot\mathbf{w}_h dV - \left(\int_{\Gamma_e}\frac{\mathcal{G}_e(\psi_{2n}^h|_{\Gamma_e}) - \mathcal{G}_e(\psi_{2n-1}^h|_{\Gamma_e})}{\tau_n}\varphi_h dA - \int_{\Gamma_e}\frac{\mathcal{G}_e(\psi_2(t_n)|_{\Gamma_e}) - \mathcal{G}_e(\psi_2(t_{n-1})|_{\Gamma_e})}{\tau_n}\varphi_h dA\right) \\ &+\frac{1}{\sigma\mu}\int_{\Omega_c}(\nabla\times\varpi^n)\cdot(\nabla\times\mathbf{w}_h)dV = -\mathbf{R}^n(\mathbf{w}_h), \quad \forall\ \mathbf{w}_h \in \mathbf{X}_h, \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} \mathbf{R}^n(\mathbf{w}_h) = \ & \int_{\Omega_b}\frac{\mathbf{P}_1(\mathbf{H}_0(t_n)) - \mathbf{P}_1(\mathbf{H}_0(t_{n-1}))}{\tau_n}\cdot\mathbf{w}_h dV - \int_{\Gamma_e}\frac{\mathcal{G}_e(\psi_2(t_n)|_{\Gamma_e}) - \mathcal{G}_e(\psi_2(t_{n-1})|_{\Gamma_e})}{\tau_n}\varphi_h dA \\ & - \int_{\Omega_b}\frac{\partial\mathbf{H}_0(t_n)}{\partial t}\cdot\mathbf{w}_h dV + \int_{\Gamma_e}\frac{\partial\mathcal{G}_e(\psi_2|_{\Gamma_e})}{\partial t}(t_n)\varphi_h dA \\ & + \frac{1}{\sigma\mu}\int_{\Omega_c}\nabla\times(\mathbf{P}_c(\mathbf{H}_0(t_n)) - \mathbf{H}_0(t_n))\cdot(\nabla\times\mathbf{w}_h)dV. \end{aligned}$$

Let

$$\begin{aligned}
\mathbf{R}_1^n(\mathbf{w}_h) = & \int_{\Omega_b}(\mathbf{P}_1-\mathbf{I})\left(\frac{\mathbf{H}_0(t_n)-\mathbf{H}_0(t_{n-1})}{\tau_n}\right)\cdot\mathbf{w}_h dV,\\
\mathbf{R}_2^n(\mathbf{w}_h) = & \int_{\Omega_b}\left(\frac{\mathbf{H}_0(t_n)-\mathbf{H}_0(t_{n-1})}{\tau_n}-\frac{\partial\mathbf{H}_0(t_n)}{\partial t}\right)\cdot\mathbf{w}_h dV-\left(\int_{\Gamma_e}\frac{\mathcal{G}_e(\psi_2(t_n)|_{\Gamma_e})-\mathcal{G}_e(\psi_2(t_{n-1})|_{\Gamma_e})}{\tau_n}\varphi_h dA\right.\\
& \left.-\int_{\Gamma_e}\frac{\partial\mathcal{G}_e(\psi_2|_{\Gamma_e})}{\partial t}(t_n)\varphi_h dA\right),\\
\mathbf{R}_3^n(\mathbf{w}_h) = & \frac{1}{\sigma\mu}\int_{\Omega_c}\nabla\times(\mathbf{P}_c(\mathbf{H}_0(t_n))-\mathbf{H}_0(t_n))\cdot(\nabla\times\mathbf{w}_h)dV,
\end{aligned}$$

so

$$\mathbf{R}^n(\mathbf{w}_h)=\mathbf{R}_1^n(\mathbf{w}_h)+\mathbf{R}_2^n(\mathbf{w}_h)+\mathbf{R}_3^n(\mathbf{w}_h).$$

Take $\mathbf{w}_h=\varpi^n$ in (5.22), then it follows that

$$\begin{aligned}
& \int_{\Omega_b}\overline{\partial}_t\varpi^n\cdot\varpi^n dV+\frac{1}{\sigma\mu}\int_{\Omega_c}(\nabla\times\varpi^n)\cdot(\nabla\times\varpi^n)dV\\
& -\frac{1}{\tau_n}\left(\int_{\Gamma_e}\mathcal{G}_e(\psi_{2n}^h|_{\Gamma_e}-\psi_2(t_n)|_{\Gamma_e})\alpha_n^h dA-\int_{\Gamma_e}\mathcal{G}_e(\psi_{2n-1}^h|_{\Gamma_e}-\psi_2(t_{n-1})|_{\Gamma_e})\alpha_n^h dA\right)\\
\leq & \ |\mathbf{R}_1^n(\varpi^n)+\mathbf{R}_2^n(\varpi^n)+\mathbf{R}_3^n(\varpi^n)|,
\end{aligned}\tag{5.23}$$

where $\alpha_n^h=\psi_{1n}^h-\psi_1(t_n)$ for $n=0,1,2,\ldots,M$.

From now on, for any $\mathbf{w}=\begin{cases}\mathbf{w}, \text{ in } \Omega_c\\ \nabla\varphi, \text{ in } \Omega_{e_1}\end{cases}\in\mathbf{X}$, we let

$$|||\mathbf{w}|||:=\left(\|\mathbf{w}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_c)}+\|\nabla\varphi\|^2_{\mathbf{L}^2(\Omega_{e_1})}\right)^{\frac{1}{2}}.$$

We can take $\alpha_n^h|_{\Gamma_e}=\psi_{2n}^h|_{\Gamma_e}-\psi_2(t_n)|_{\Gamma_e}$ in (5.23) for $n=0,1,2,\ldots,M$, then it is obtained that

$$\begin{aligned}
& |||\varpi^n|||+\|\psi_{2n}^h-\psi_2(t_n)\|_{H^{\frac{1}{2}}(\Gamma_e)}\\
\leq & \ C\left(|||\varpi^{n-1}|||+\|\psi_{2n-1}^h-\psi_2(t_{n-1})\|_{H^{\frac{1}{2}}(\Gamma_e)}\right.\\
& +\tau_n\|\frac{\mathbf{H}_0(t_n)-\mathbf{H}_0(t_{n-1})}{\tau_n}-\frac{\partial\mathbf{H}_0}{\partial t}(t_n)\|_{\mathbf{L}^2(\Omega_b)}+\tau_n\|\frac{\psi_2(t_n)-\psi_2(t_{n-1})}{\tau_n}-\frac{\partial\psi_2}{\partial t}(t_n)\|_{H^{\frac{1}{2}}(\Gamma_e)}\\
& \left.+\tau_n\|(\mathbf{P}_1-\mathbf{I})\left(\frac{\mathbf{H}_0(t_n)-\mathbf{H}_0(t_{n-1})}{\tau_n}\right)\|_{L^2(\Omega_b)}+\tau_n\|\nabla\times(\mathbf{P}_c(\mathbf{H}_0(t_n))-\mathbf{H}_0(t_n))\|_{\mathbf{L}^2(\Omega_c)}\right).
\end{aligned}$$

By using recursion, it follows that

$$\begin{aligned}
& |||\varpi^n|||+\|\psi_{2n}^h-\psi_2(t_n)\|_{H^{\frac{1}{2}}(\Gamma_e)}\\
\leq & \ C\sum_{j=1}^n\tau_j\left(\|\nabla\times(\mathbf{P}_c(\mathbf{H}_0(t_i))-\mathbf{H}_0(t_i))\|_{\mathbf{L}^2(\Omega_c)}+\|\frac{\mathbf{H}_0(t_i)-\mathbf{H}_0(t_{i-1})}{\tau_i}-\frac{\partial\mathbf{H}_0}{\partial t}(t_i)\|_{\mathbf{L}^2(\Omega_b)}\right.\\
& \left.+\|\frac{\psi_2(t_i)-\psi_2(t_{i-1})}{\tau_i}-\frac{\partial\psi_2}{\partial t}(t_i)\|_{H^{\frac{1}{2}}(\Gamma_e)}+\|(\mathbf{P}_1-\mathbf{I})\left(\frac{\mathbf{H}_0(t_i)-\mathbf{H}_0(t_{i-1})}{\tau_i}\right)\|_{L^2(\Omega_b)}\right).
\end{aligned}$$

Since

$$(\mathbf{P}_1-\mathbf{I})\left(\frac{\mathbf{H}_0(t_j)-\mathbf{H}_0(t_{j-1})}{\tau_j}\right)=(\mathbf{P}_1-\mathbf{I})\left(\tau_j^{-1}\int_{t_{j-1}}^{t_j}\mathbf{H}_{0t}(l)dl\right)=\tau_j^{-1}\int_{t_{j-1}}^{t_j}(\mathbf{P}_1-\mathbf{I})(\mathbf{H}_{0t}(l))dl,$$

we have

$$\begin{aligned}
& \sum_{j=1}^n\tau_j\|(\mathbf{P}_1-\mathbf{I})\left(\frac{\mathbf{H}_0(t_j)-\mathbf{H}_0(t_{j-1})}{\tau_j}\right)\|_{L^2(\Omega_b)}\\
\leq & \sum_{j=1}^n\int_{t_{j-1}}^{t_j}C\left(h^s\|\mathbf{H}_{0t}(l)\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)}+h^{r-1}\|\psi_{1t}(l)\|_{H^r(\Omega_{e_1})/P_0}\right)dl\\
\leq & \ C\left(h^s\int_0^{t_n}\|\mathbf{H}_{0t}(l)\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)}dl+h^{r-1}\int_0^{t_n}\|\psi_{1t}(l)\|_{H^r(\Omega_{e_1})/P_0}dl\right).
\end{aligned}$$

Moreover, it is known that

$$\frac{\mathbf{H}_0(t_j)-\mathbf{H}_0(t_{j-1})}{\tau_j}-\frac{\partial \mathbf{H}_0(t_j)}{\partial t}=-\tau_j^{-1}\int_{t_{j-1}}^{t_j}(l-t_{j-1})\mathbf{H}_{0tt}(l)dl,$$

and

$$\frac{\psi_2(t_j)-\psi_2(t_{j-1})}{\tau_j}-\frac{\partial \psi_2(t_j)}{\partial t}=-\tau_j^{-1}\int_{t_{j-1}}^{t_j}(l-t_{j-1})\psi_{2tt}(l)dl,$$

then we have

$$\begin{aligned}&\sum_{j=1}^{n}\tau_j\|\left(\tfrac{\mathbf{H}_0(t_j)-\mathbf{H}_0(t_{j-1})}{\tau_j}-\tfrac{\partial \mathbf{H}_0(t_j)}{\partial t}\right)\|_{\mathbf{L}^2(\Omega_b)}\\ &\leq C\sum_{j=1}^{n}\|\textstyle\int_{t_{j-1}}^{t_j}(l-t_{j-1})\mathbf{H}_{0tt}(l)dl\|_{\mathbf{L}^2(\Omega_b)}\\ &\leq C\tau\textstyle\int_0^{t_n}\|\mathbf{H}_{0tt}(l)\|_{\mathbf{L}^2(\Omega_b)}dl,\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{n}\tau_j\|\tfrac{\psi_2(t_i)-\psi_2(t_{i-1})}{\tau_i}-\tfrac{\partial\psi_2}{\partial t}(t_i)\|_{H^{\frac{1}{2}}(\Gamma_e)}\\ &\leq C\sum_{j=1}^{n}\|\textstyle\int_{t_{j-1}}^{t_j}(l-t_{j-1})\psi_{2tt}(l)dl\|_{H^{\frac{1}{2}}(\Gamma_e)}\\ &\leq C\tau\textstyle\int_0^{t_n}\|\psi_{2tt}(l)\|_{H^{\frac{1}{2}}(\Gamma_e)}dl.\end{aligned}$$

In addition,

$$\sum_{j=1}^{n}\tau_j\|\nabla\times(\mathbf{P}_c(\mathbf{H}_0(t_i))-\mathbf{H}_0(t_i))\|_{\mathbf{L}^2(\Omega_c)}\leq Ct_nh^s\textstyle\int_0^{t_n}\|\mathbf{H}_{0t}(l)\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)}dl.$$

Therefore, we have

$$\begin{aligned}&|||\varpi^n|||+\|\psi_{2n}^h-\psi_2(t_n)\|_{H^{\frac{1}{2}}(\Gamma_e)}\\ \leq\;& C\Big(h^s(1+t_n)\textstyle\int_0^{t_n}\|\mathbf{H}_{0t}(l)\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)}dl+h^{r-1}\int_0^{t_n}\|\psi_{1t}(l)\|_{H^r(\Omega_{e_1})/P_0}dl\\ &+\tau\textstyle\int_0^{t_n}\|\psi_{2tt}(l)\|_{H^{\frac{1}{2}}(\Gamma_e)}dl+\tau\int_0^{t_n}\|\mathbf{H}_{0tt}(l)\|_{\mathbf{L}^2(\Omega_b)}dl\Big).\end{aligned}$$

By using the triangle inequality and (5.21), we get

$$\begin{aligned}&\|\mathbf{H}_{0n}^h-\mathbf{H}_0(t_n)\|_{\mathbf{H}(\mathbf{curl},\Omega_c)}+\|\nabla\psi_{1n}^h-\nabla\psi_1(t_n)\|_{\mathbf{L}^2(\Omega_{e_1})}+\|\psi_{2n}^h-\psi_2(t_n)\|_{H^{\frac{1}{2}}(\Gamma_e)}\\ \leq\;& C\Big(h^s(1+t_n)\textstyle\int_0^{t_n}\|\mathbf{H}_{0t}(l)\|_{\mathbf{H}^s(\mathbf{curl},\Omega_c)}dl+h^{r-1}\int_0^{t_n}\|\psi_{1t}(l)\|_{H^r(\Omega_{e_1})/P_0}dl\\ &+\tau\textstyle\int_0^{t_n}\|\psi_{2tt}(l)\|_{H^{\frac{1}{2}}(\Gamma_e)}dl+\tau\int_0^{t_n}\|\mathbf{H}_{0tt}(l)\|_{\mathbf{L}^2(\Omega_b)}dl\Big),\quad \text{for } n=1,2,\ldots,M.\qquad\square\end{aligned}$$

Theorem 5.3. *If $\{\mathbf{H}_{0n}^{\dagger h}\}$ and $\{\mathbf{H}_{0n}^h\}$ are the solutions of (5.17) and (5.16) respectively, then*

$$\begin{aligned}&\|\mathbf{H}_{0n}^h-\mathbf{H}_{0n}^{\dagger h}\|_{\mathbf{H}(\mathbf{curl},\Omega_c)}+\|\nabla\psi_{1n}^h-\nabla\psi_{1n}^{\dagger h}\|_{\mathbf{L}^2(\Omega_{e_1})}+\|\psi_{2n}^h|_{\Gamma_e}-\psi_{2n}^{\dagger h}|_{\Gamma_e}\|_{H^{\frac{1}{2}}(\Gamma_e)}\\ \leq\;& C\sum_{k=1}^{n-1}\Big(\|\mathbf{H}_{0k}^{\dagger h}-\widetilde{\mathbf{H}}_{0k}^h\|_{\mathbf{L}^2(\Omega_c)}+\|\nabla\psi_{1k}^{\dagger h}-\nabla\widetilde{\psi}_{1k}^h\|_{\mathbf{L}^2(\Omega_{e_1})}+\|\psi_{2k}^{\dagger h}|_{\Gamma_e}-\widetilde{\psi}_{2k}^h|_{\Gamma_e}\|_{H^{\frac{1}{2}}(\Gamma_e)}\Big).\end{aligned}\tag{5.24}$$

Proof: subtracting (5.17) from (5.16), it can be obtained that

$$\begin{aligned}&\textstyle\int_{\Omega_c}(\frac{\mathbf{H}_{0n}^h-\mathbf{H}_{0n}^{\dagger h}}{\tau_n}-\frac{\mathbf{H}_{0n-1}^h-\widetilde{\mathbf{H}}_{0n-1}^h}{\tau_n})\cdot\mathbf{w}_hdV-\int_{\Gamma_e}(\frac{\mathcal{G}_e(\psi_{2n}^h|_{\Gamma_e})-\mathcal{G}_e(\psi_{2n}^{\dagger h}|_{\Gamma_e})}{\tau_n}-\frac{\mathcal{G}_e(\psi_{2n-1}^h|_{\Gamma_e})-\mathcal{G}_e(\widetilde{\psi}_{2n-1}^h|_{\Gamma_e})}{\tau_n})\varphi_hdA\\ &+\tfrac{1}{\sigma\mu}\textstyle\int_{\Omega_c}(\nabla\times(\mathbf{H}_{0n}^h-\mathbf{H}_{0n}^{\dagger h}))\cdot(\nabla\times\mathbf{w}_h)dV+\int_{\Omega_{e_1}}(\frac{\nabla\psi_{1n}^h-\nabla\psi_{1n}^{\dagger h}}{\tau_n}-\frac{\nabla\psi_{1n-1}^h-\nabla\widetilde{\psi}_{1n-1}^h}{\tau_n})\cdot\nabla\varphi_hdV=0,\end{aligned}$$

for any $\mathbf{w}_h \in \mathbf{X}_h$, so we get

$$\begin{aligned} & \int_{\Omega_c} (\frac{\mathbf{H}_{0n}^h - \mathbf{H}_{0n}^{\dagger h}}{\tau_n}) \cdot \mathbf{w}_h dV + \frac{1}{\sigma\mu} \int_{\Omega_c} (\nabla \times (\mathbf{H}_{0n}^h - \mathbf{H}_{0n}^{\dagger h})) \cdot (\nabla \times \mathbf{w}_h) dV \\ & + \int_{\Omega_{e_1}} (\frac{\nabla\psi_{1n}^h - \nabla\psi_{1n}^{\dagger h}}{\tau_n}) \cdot \nabla\varphi_h dV - \int_{\Gamma_e} (\frac{\mathcal{G}_e(\psi_{2n}^h|_{\Gamma_e}) - \mathcal{G}_e(\psi_{2n}^{\dagger h}|_{\Gamma_e})}{\tau_n}) \varphi_h dA \\ = & \int_{\Omega_c} (\frac{\mathbf{H}_{0n-1}^h - \widetilde{\mathbf{H}}_{0n-1}^h}{\tau_n}) \cdot \mathbf{w}_h dV + \int_{\Omega_{e_1}} (\frac{\nabla\psi_{1n-1}^h - \nabla\widetilde{\psi}_{1n-1}^h}{\tau_n}) \cdot \nabla\varphi_h dV \\ & - \int_{\Gamma_e} (\frac{\mathcal{G}_e(\psi_{2n-1}^h|_{\Gamma_e}) - \mathcal{G}_e(\widetilde{\psi}_{2n-1}^h|_{\Gamma_e})}{\tau_n}) \varphi_h dA, \qquad \forall\ \mathbf{w}_h \in \mathbf{X}_h. \end{aligned} \tag{5.25}$$

Let

$$I_n = (\|\mathbf{H}_{0n}^h - \mathbf{H}_{0n}^{\dagger h}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_c)} + \|\nabla\psi_{1n}^h - \nabla\psi_{1n}^{\dagger h}\|^2_{\mathbf{L}^2(\Omega_1)} + \|\psi_{2n}^h|_{\Gamma_e} - \psi_{2n}^{\dagger h}|_{\Gamma_e}\|^2_{H^{\frac{1}{2}}(\Gamma_e)})^{\frac{1}{2}}.$$

By taking $\mathbf{w}_h = (\mathbf{H}_{0n}^h - \mathbf{H}_{0n}^{\dagger h})$ in (5.25), we have

$$\begin{aligned} I_n^2 \leq\ & C\Big(\|\mathbf{H}_{0n-1}^h - \widetilde{\mathbf{H}}_{0n-1}^h\|_{\mathbf{L}^2(\Omega_c)} \|\mathbf{H}_{0n}^h - \mathbf{H}_{0n}^{\dagger h}\|_{\mathbf{L}^2(\Omega_c)} \\ & + \|\nabla\psi_{1n-1}^h - \nabla\widetilde{\psi}_{1n-1}^h\|_{\mathbf{L}^2(\Omega_{e_1})} \|\nabla\psi_{1n}^h - \nabla\psi_{1n}^{\dagger h}\|_{\mathbf{L}^2(\Omega_{e_1})} \\ & + \|\psi_{2n-1}^h|_{\Gamma_e} - \widetilde{\psi}_{2n-1}^h|_{\Gamma_e}\|_{H^{\frac{1}{2}}(\Gamma_e)} \|\psi_{2n}^h|_{\Gamma_e} - \psi_{2n}^{\dagger h}\|_{H^{\frac{1}{2}}(\Gamma_e)}\Big) \\ \leq\ & C I_n \Big(\|\mathbf{H}_{0n-1}^h - \widetilde{\mathbf{H}}_{0n-1}^h\|^2_{\mathbf{L}^2(\Omega_c)} + \|\nabla\psi_{1n-1}^h - \nabla\widetilde{\psi}_{1n-1}^h\|^2_{\mathbf{L}^2(\Omega_{e_1})} \\ & + \|\psi_{2n-1}^h|_{\Gamma_e} - \widetilde{\psi}_{2n-1}^h|_{\Gamma_e}\|^2_{H^{\frac{1}{2}}(\Gamma_e)}\Big)^{\frac{1}{2}}, \end{aligned}$$

so

$$\begin{aligned} I_n \leq C\Big(& \|\mathbf{H}_{0n-1}^h - \widetilde{\mathbf{H}}_{0n-1}^h\|^2_{\mathbf{L}^2(\Omega_c)} + \|\nabla\psi_{1n-1}^h - \nabla\widetilde{\psi}_{1n-1}^h\|^2_{\mathbf{L}^2(\Omega_{e_1})} \\ & + \|\psi_{2n-1}^h|_{\Gamma_e} - \widetilde{\psi}_{2n-1}^h|_{\Gamma_e}\|^2_{H^{\frac{1}{2}}(\Gamma_e)}\Big)^{\frac{1}{2}}. \end{aligned}$$

By recursion and the triangle inequality, we can get

$$\begin{aligned} & \|\mathbf{H}_{0n}^h - \mathbf{H}_{0n}^{\dagger h}\|_{\mathbf{H}(\mathbf{curl},\Omega_c)} + \|\nabla\psi_{1n}^h - \nabla\psi_{1n}^{\dagger h}\|_{\mathbf{L}^2(\Omega_{e_1})} + \|\psi_{2n}^h|_{\Gamma_e} - \psi_{2n}^{\dagger h}|_{\Gamma_e}\|_{H^{\frac{1}{2}}(\Gamma_e)} \\ \leq\ & C\sum_{k=1}^{n-1}\Big(\|\mathbf{H}_{0k}^{\dagger h} - \widetilde{\mathbf{H}}_{0k}^h\|_{\mathbf{L}^2(\Omega_c)} + \|\nabla\psi_{1k}^{\dagger h} - \nabla\widetilde{\psi}_{1k}^h\|_{\mathbf{L}^2(\Omega_{e_1})} + \|\psi_{2k}^{\dagger h}|_{\Gamma_e} - \widetilde{\psi}_{2k}^h|_{\Gamma_e}\|_{H^{\frac{1}{2}}(\Gamma_e)}\Big), \end{aligned}$$

which completes the proof. $\square$

Theorem 5.1 shows that the right hand side of (5.24) is convergent to zero, then Theorems 5.2 and 5.3 imply that the backward Euler-DtN alternating algorithm is convergent.

6 Implementation

In this section, we will discuss some implementation details of the backward Euler-DtN alternating algorithm.

Bases of several spaces.

1. For any $K \in \mathcal{T}_{h_c}^c$, we define

$$\mathcal{C}_c(K) = \{e \mid e \text{ is the edge of } K\},$$

so the set $\mathcal{E}_h = \bigcup_{K \in \mathcal{T}_{h_c}^c} \mathcal{C}_c(K)$. Let the number of edges in $\mathcal{E}_h$ be M_c.

For any $e \in \mathcal{E}_h$, there is a relevant function $\mathbf{u}_{h,e} \in N_{1,h}^1$ whose support is in $\mathcal{F}_e = \bigcup_{K \in \mathcal{Z}(e)} K$. Here

$$\mathcal{Z}(e) = \{K \mid e \in \mathcal{C}_c(K)\}.$$

Moreover, $\mathbf{u}_{h,e}$ satisfies that

$$\int_{e_i} \mathbf{u}_{h,e} \cdot \tau_{e_i} ds = \begin{cases} 1, & \text{if } e_i = e, \\ 0, & \text{if } e_i \neq e, \end{cases}$$

for any $e_i \in \mathcal{E}_h$. So we get a basis of $N_{1,h}^1$, denoted by

$$\mathcal{B}_c = \{\mathbf{u}_{h,e_i} \in N_{1,h}^1 \mid e_i \in \mathcal{E}_h, i = 1, 2, \ldots, M_c\}.$$

2. For any $K \in \mathcal{T}_{h_e}^e$, we define

$$\mathcal{C}_e(K) = \{nd \mid nd \text{ is the vertex of } K\},$$

so the set $\mathcal{N}_h = \bigcup_{K \in \mathcal{T}_{h_e}^e} \mathcal{C}_e(K)$. Let the number of nodes in $\mathcal{N}_h$ be N_e.

For any $nd \in \mathcal{N}_h$, there is a relevant function $\varphi_{h,nd} \in S_h$ whose support is in $\mathcal{F}_{nd} = \bigcup_{K \in \mathcal{Y}(nd)} K$. Here

$$\mathcal{Y}(nd) = \{K \mid nd \in \mathcal{C}_e(K)\}.$$

Moreover, $\varphi_{h,nd}$ satisfies that

$$\varphi_{h,nd}(nd_i) = \begin{cases} 1, & \text{if } nd_i = nd, \\ 0, & \text{if } nd_i \neq nd, \end{cases}$$

for any $nd_i \in \mathcal{N}_h$. So we get a basis of S_h, denoted by

$$\mathcal{B}_e = \{\varphi_{h,nd_i} \in S_h \mid nd_i \in \mathcal{N}_h, i = 1, 2, \ldots, N_e\}.$$

3. According to the relationship between S_h and $\widetilde{V}_{2h}$, we can also obtain a basis of $\widetilde{V}_{2h}$,

$$\widetilde{\mathcal{B}}_e = \{\widetilde{\varphi}_{h,nd_i} \mid \widetilde{\varphi}_{h,nd_i} \circ \widehat{F}_h = \varphi_{h,nd_i},\ nd_i \in \mathcal{N}_h, i = 1, 2, \ldots, N_e\}.$$

4. Let the number of nodes of $\mathcal{T}_{h_e}^e$ on the spherical artificial boundary be N_{s1} and the number of nodes on Γ_c be N_{s2}. It is easy to see that $(N_{s1} + N_{s2}) < N_e$. Define

$$\mathcal{N}_{s1} = \{nd \in \mathcal{N}_h \mid nd \text{ is a node on the spherical artificial boundary}\},$$

$$\mathcal{N}_{s2} = \{nd \in \mathcal{N}_h \mid nd \text{ is a node on } \Gamma_c\},$$

and

$$\mathcal{N}_s = \mathcal{N}_{s1} \cup \mathcal{N}_{s2}.$$

For any $nd \in \mathcal{N}_s$, there is a corresponding function $\widetilde{\varphi}_{h,nd} \in \widetilde{\mathcal{B}}_e$. So we can get the basis of χ_h, denoted by

$$\mathcal{B}_s = \{\chi_i \mid \chi_i = \widetilde{\varphi}_{h,nd_{s(i)}}|_{\partial\Omega_{e_1}},\ nd_{s(i)} \in \mathcal{N}_s,\ (nd_{s(i)} \in \mathcal{N}_{s1},\ \text{if } 1 \leq i \leq N_{s1})\},$$

where $nd_{s(i)}$ is the ith node in $\mathcal{N}_s$ and $s(i)$th node in $\mathcal{N}_h$.

Expansion of χ_i.

According to the definition of $\mathcal{B}_s$, χ_i, $i = 1, \ldots, N_{s1}$, correspond to the nodes on Γ_e, which can be expanded as:

$$\chi_i = \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l} (\chi_{i,1l}^m Y_{1l}^m(\theta,\phi) + \chi_{i,2l}^m Y_{2l}^m(\theta,\phi)) + \chi_{i,0l}^0 Y_{0l}^0(\theta,\phi) \right), \quad \text{for } i = 1, \ldots, N_{s1}, \tag{6.1}$$

where

$$\begin{cases} \chi_{i,0l}^0 = & \frac{1}{R^2} \int_{\Gamma_e} \chi_i(\theta,\phi) Y_{0l}^0(\theta,\phi) dA, \\ \chi_{i,1l}^m = & \frac{1}{R^2} \int_{\Gamma_e} \chi_i(\theta,\phi) Y_{1l}^m(\theta,\phi) dA, \\ \chi_{i,2l}^m = & \frac{1}{R^2} \int_{\Gamma_e} \chi_i(\theta,\phi) Y_{2l}^m(\theta,\phi) dA. \end{cases}$$

The computation of $\mathcal{G}_e(\lambda_{n,m}^e|_{\Gamma_e})$.

Since $\lambda_{n,m}^e|_{\Gamma_e}$ has the expansion:

$$\lambda_{n,m}^e|_{\Gamma_e} = \sum_{k=1}^{N_{s1}} q_{en,k}^m \chi_k, \tag{6.2}$$

we can get the series form of $\lambda_{n,m}^e|_{\Gamma_e}$ with (6.1). Let the approximate value of $\mathcal{G}_e(\lambda_{n,m}^e|_{\Gamma_e})$ in χ_h be $\sum_{k=1}^{N_{s1}} \mu_k \chi_k$, where μ_k , $k = 1, \ldots, N_{s1}$, are undetermined. Then it follows that

$$\sum_{k=1}^{N_{s1}} \mu_k \chi_k = \sum_{k=1}^{N_{s1}} q_{en,k}^m \mathcal{G}_e \chi_k.$$

It means that

$$\sum_{k=1}^{N_{s1}} \mu_k \langle \chi_k, \chi_j \rangle_{H^{-\frac{1}{2}}(\Gamma_e)} = \sum_{k=1}^{N_{s1}} q_{en,k}^m \langle \mathcal{G}_e \chi_k, \chi_j \rangle_{H^{-\frac{1}{2}}(\Gamma_e)}, \quad \text{for j} = 1, \ldots, \text{N}_{s1}.$$

This leads to the following system:

$$A^\top X = Q^\top Y, \tag{6.3}$$

where A is a symmetrical matrix with $A_{ij} = \langle \chi_i, \chi_j \rangle_{H^{-\frac{1}{2}}(\Gamma_e)}$, $Q_{ij} = \langle \mathcal{G}_e \chi_i, \chi_j \rangle_{H^{-\frac{1}{2}}(\Gamma_e)}$ for $i, j = 1, \ldots, N_{s1}$, $X = (\mu_1, \cdots, \mu_{N_{s1}})^\top$ and $Y = (q_{en,1}^m, \cdots, q_{en,N_{s1}}^m)^\top$. Therefore, $\mathcal{G}_e(\lambda_{n,m}^e|_{\Gamma_e})$ can be obtained by solving the system (6.3).

Let

$$\chi_i = \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l} (\chi_{i,1l}^m Y_{1l}^m(\theta,\phi) + \chi_{i,2l}^m Y_{2l}^m(\theta,\phi)) + \chi_{i,0l}^0 Y_{0l}^0(\theta,\phi) \right)$$

and

$$\chi_j = \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l} (\chi_{j,1l}^m Y_{1l}^m(\theta,\phi) + \chi_{j,2l}^m Y_{2l}^m(\theta,\phi)) + \chi_{j,0l}^0 Y_{0l}^0(\theta,\phi) \right),$$

by using the definition of the hermitian product in $H^{-\frac{1}{2}}(\Gamma_e)$, then we can get

$$\langle \chi_i, \chi_j \rangle_{H^{-\frac{1}{2}}(\Gamma_e)} = R^2 \sum_{l=0}^{\infty} (l+1)^{-1} \left(\sum_{m=1}^{l} (\chi_{i,1l}^m \chi_{j,1l}^m + \chi_{i,2l}^m \chi_{j,2l}^m) + \chi_{i,0l}^m \chi_{j,0l}^m \right),$$

and

$$\langle \mathcal{G}_e \chi_i, \chi_j \rangle_{H^{-\frac{1}{2}}(\Gamma_e)} = \sum_{l=0}^{\infty} (-R) \left(\sum_{m=1}^{l} (\chi_{i,1l}^m \chi_{j,1l}^m + \chi_{i,2l}^m \chi_{j,2l}^m) + \chi_{i,0l}^m \chi_{j,0l}^m \right).$$

It is clear that the two infinite series can not be used directly to compute. In applications, we need to substitute the whole infinite series by the sum of the first N terms. Therefore, the feasibility of the substitution should be discussed. For any $1 \le i, j \le N_{s1}$, we have the following error estimates:

$$\begin{aligned} \mathcal{L}_{ij} &= \left| \langle \chi_i, \chi_j \rangle_{H^{-\frac{1}{2}}(\Gamma_e)} - R^2 \sum_{l=0}^{N} (l+1)^{-1} \left(\sum_{m=1}^{l} (\chi_{i,1l}^m \chi_{j,1l}^m + \chi_{i,2l}^m \chi_{j,2l}^m) + \chi_{i,0l}^m \chi_{j,0l}^m \right) \right| \\ &= R^2 \left| \sum_{l=N+1}^{\infty} (l+1)^{-1} \left(\sum_{m=1}^{l} (\chi_{i,1l}^m \chi_{j,1l}^m + \chi_{i,2l}^m \chi_{j,2l}^m) + \chi_{i,0l}^m \chi_{j,0l}^m \right) \right| \\ &\le R^2 (N+1)^{-2} \sum_{l=0}^{\infty} (l+1) \left(\sum_{m=1}^{l} (|\chi_{i,1l}^m||\chi_{j,1l}^m| + |\chi_{i,2l}^m||\chi_{j,2l}^m|) + |\chi_{i,0l}^m||\chi_{j,0l}^m| \right). \end{aligned}$$

By Cauchy inequality, we have

$$\mathcal{L}_{ij} \le (N+1)^{-2} \|\chi_i\|_{H^{\frac{1}{2}}(\Gamma_e)} \|\chi_j\|_{H^{\frac{1}{2}}(\Gamma_e)} \le C N^{-2} \|\chi_i\|_{H^{\frac{1}{2}}(\Gamma_e)} \|\chi_j\|_{H^{\frac{1}{2}}(\Gamma_e)}.$$

Similarly, it is obtained that

$$\begin{aligned} &\left| \langle \mathcal{G}_e \chi_i, \chi_j \rangle_{H^{-\frac{1}{2}}(\Gamma_e)} - \sum_{l=0}^{N} (-R) \left(\sum_{m=1}^{l} (\chi_{i,1l}^m \chi_{j,1l}^m + \chi_{i,2l}^m \chi_{j,2l}^m) + \chi_{i,0l}^m \chi_{j,0l}^m \right) \right| \\ &\le C N^{-1} \|\chi_i\|_{H^{\frac{1}{2}}(\Gamma_e)} \|\chi_j\|_{H^{\frac{1}{2}}(\Gamma_e)}, \end{aligned}$$

where C is a generic constant. From the previous estimates, we know that it is feasible to substitute the sum of the first N terms for the infinite series in practice.

7 Numerical examples

In this section, we will give some numerical examples for the backward Euler-DtN alternating algorithm. Let the cube

$$\Omega_c = \{(x, y, z) \mid -a \le x \le a, -a \le y \le a, -a \le z \le a, a > 0\}$$

and the boundary of the cube be Γ_c. We assume that Ω_e is the complement of the cube and the artificial boundary is the sphere

$$\Gamma_e = \{(x, y, z) \mid r = R_1, r = \sqrt{x^2 + y^2 + z^2}\}.$$

For the bounded domain

$$\Omega_{e_1} = \{(x, y, z) \in \Omega_e \mid r < R_1, r = \sqrt{x^2 + y^2 + z^2}\}$$

and Ω_c, we all use the tetrahedral meshes and the two meshes match on Γ_c. In step 2 of the backward Euler-DtN alternating algorithm, we substitute the sum of the first 10 terms for the infinite series.

Example 1. In this numerical example, let $a = 2$, $R_1 = 6$, $\mathbf{H}_s = \cos(\mathbf{x} + \mathbf{y} + \mathbf{z} + t)(1, 1, 1)^\top$ and the time interval be $[0, 1]$. Assume that $M = 3$ and $\tau_i = \frac{1}{3}$ for $i = 1, 2, 3$. We define

$$e^h(n, m) = \|\Phi_n^h - S_n^h \lambda_{n,m}\|_{L^2(\partial\Omega_{e_1})},$$

which is the residual of the mth iteration at t_n, so the stop condition of the iteration at t_n can be $e^h(n, m) < 10^{-4}$. Since the exact solution of this example is not known, we only give the errors

$$Error_c(n, m) = \|\mathbf{H}_{0n,m+1}^h - \mathbf{H}_{0n,m}^h\|_{\mathbf{L}^2(\Omega_c)}$$

and

$$Error_e(n, m) = \|\nabla\psi_{1n,m+1}^h - \nabla\psi_{1n,m}^h\|_{L^2(\Omega_{e_1})}.$$

We assume that $e^h(n)$, $Error_c(n)$ and $Error_e(n)$ are equal to $e^h(n, m)$, $Error_c(n, m)$ and $Error_e(n, m)$ respectively, when the stop condition of the iteration at t_n is satisfied. Let the number of the nodes in each direction in Ω_c be n_c and the number of the nodes in each direction in Ω_{e_1} be n_e. The number of iterations at t_n is denoted by ITE_n. The initial value $\lambda_{1,0}$ is 0 and the initial value $\lambda_{i,0}$ is the numerical result of the last time step for $i = 2, 3$. For convenience, we let all the relaxation factor $\vartheta_{n,m} = 0.5$. Then the numerical results are shown in table 1.1.

Table 1.1

(Computational errors in different meshes for each time step)

t_n	n_c	n_e	ITE_n	$e^h(n)$	$Error_c(n)$	$Error_e(n)$
$\frac{1}{3}$	3	3	57	8.89091e-005	9.6626e-007	1.1343e-05
	5	5	24	8.91014e-005	4.78344e-006	4.31362e-005
	9	9	20	7.62595e-005	1.56637e-005	0.000133394
	17	17	21	9.83177e-005	1.69839e-005	0.000126887
$\frac{2}{3}$	3	3	35	8.31169e-005	9.5219e-007	1.06897e-005
	5	5	21	7.23104e-005	3.87084e-006	3.49387e-005
	9	9	21	6.71779e-005	1.37622e-005	0.00011765
	17	17	23	8.40778e-005	1.45327e-005	0.000108504
1	3	3	35	9.26529e-005	1.0298e-006	1.18596e-005
	5	5	22	6.67319e-005	3.61168e-006	3.23214e-005
	9	9	21	9.3471e-005	1.91476e-005	0.0001637
	17	17	24	7.81718e-005	1.35124e-005	0.000100882

The table shows that the algorithm is convergent at each time step. Since the exact solution of this example is not known, we can not get the order of convergence of the algorithm from the numerical results.

Example 2. This example will compare the convergence rates for different relaxation factors at each time step. We also adopt the numerical example introduced in Example 1 and only consider the case that $n_c = n_e = 5$. Let the relaxation factors at t_n be ϑ_n. The results are listed in table 2.1-2.3.

Table 2.1

(Comparisons of convergence rates for different relaxation factors at t_1)

ϑ_1	0.1	0.2	0.3	0.4	0.5
ITE_1	104	50	31	22	24

where $e^h(1) < 1.0e - 4$.

Table 2.2

(Comparisons of convergence rates for different relaxation factors at t_2)

ϑ_2	0.1	0.2	0.3	0.4	0.5
ITE_2	100	48	30	21	21

where $e^h(2) < 1.0e - 4$.

Table 2.3

(Comparisons of convergence rates for different relaxation factors at t_3)

ϑ_3	0.1	0.2	0.3	0.4	0.5
ITE_3	98	46	29	21	22

where $e^h(3) < 1.0e - 4$.

The numerical results in example 2 indicate that the choice of the relaxation factor is important for the DtN alternating method at each time step. Moreover, if we choose a good relaxation factor, the convergence rate is not sensitive when we change the relaxation factor in its neighborhood.

8 Conclusion

In the last section, we summarize the backward Euler-DtN alternating algorithm.

1. The finite element problem and boundary element problem can be handled independently from each other at each time step.

2. At each time step, by using natural boundary reduction, the problem in unbounded domain can be converted to the one on the artificial spherical boundary and no mesh is needed in the exterior of artificial boundary. Once we get approximate solution on the artificial boundary, the approximate solution in the exterior of artificial boundary can be obtained by using (4.11).

3. In this iterative method, the action of the boundary operators can be implemented by calculating a series of spherical harmonics, then there is no need to solve integral equations.

4. The DtN alternating method at each time step converges with a rate independent of the mesh size.

5. Provided that we choose an appropriate relaxation factor, the algorithm is convergent in a few iterations at each time step.

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