
Boundary Value Problems for Differential Equations of Fractional Order

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Abstract. We carry out spectral analysis of one class of integral operators associated with fractional order differential equations that arise in mechanics. We establish a connection between the eigenvalues of these operators and the zeros of Mittag-Leffler type functions. We give sufficient conditions for complete nonselfadjointness and completeness of systems of the eigenvalues. We prove the existence and uniqueness of the solutions for several kinds of two-point boundary value problems of fractional differential equations with Caputo's derivatives or Riemann-Liouville Derivatives, and design the single shooting methods to solve them numerically.

Chapter 1. Regional problems for differential equations of fractional order

1. The basic concepts

The spectral analysis of operators of the form

$$A_{\gamma}^{[\alpha,\beta]}u(x) = c_{\alpha}\int\limits_{0}^{x}(x-t)^{\frac{1}{\alpha}-1}u(t)dt + c_{\beta,\gamma}\int\limits_{0}^{1}x^{\frac{1}{\beta}-1}(1-t)^{\frac{1}{\gamma}-1}u(t)dt$$

was carried out in [1] (similar operators were considered by G.M. Gubreev in [2]). Here α, β, γ, c_α, $c_{\beta,\gamma}$ are real numbers, and α, β, γ are positive. These operators arise in the study of boundary value problems for differential equations of fractional order (see [3] and references therein, where the corresponding Green functions are constructed).

The present paper is devoted to studying the boundary value problems for differential equations of fractional order and the accompanying integral operators of the form $A_\gamma^{[\alpha;\beta]}$.

In order to state the problems in concern, we must mention some concepts in fractional calculus.

Let $f(x) \in L_1(0,1)$. Then, the function

$$\frac{d^{-\alpha}}{dx^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-t)^{\alpha-1} f(t)dt \in L_1(0,1)$$

is called the fractional integral of order $\alpha > 0$ with starting point $x = 0$, and the function

$$\frac{d^{-\alpha}}{d(1-x)^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int\limits_x^1 (t-x)^{\alpha-1} f(t)dt \in L_1(0,1)$$

is called the fractional integral of order $\alpha > 0$ with ending point $x = 1$ (refer to [6]). Here $\Gamma(\alpha)$ is Euler's Gamma-function. It is clear that when $\alpha = 0$, we identify both of the fractional integrals with the function $f(x)$. As we know (see [6]), the function $g(x) \in L_1(0,1)$ is called the fractional derivative of the function $f(x) \in L_1(0,1)$ of order $\alpha > 0$ with starting point $x = 0$, if

$$f(x) = \frac{d^{-\alpha}}{dx^{-\alpha}} g(x).$$

Then denoting

$$g(x) = \frac{d^{\alpha}}{dx^{\alpha}} f(x),$$

we shall mean in the future, by

$$\frac{d^{\alpha}}{dx^{\alpha}},$$

the fractional integral when $\alpha < 0$ and, the fractional derivative when $\alpha > 0$. The fractional derivative

$$\frac{d^{\alpha}}{d(1-x)^{\alpha}}$$

of the function $f(x) \in L_1(0,1)$ of order $\alpha > 0$, with ending point $x = 1$ is defined in a similar way.

Let $\{\gamma_k\}_0^n$ be any set of real numbers, satisfying the condition $0 < \gamma_j \leq 1$, $(0 \leq j \leq n)$. We denote

$$\sigma_k = \sum_{j=0}^{k} \gamma_j - 1; \quad \mu_k = \sigma_k + 1 = \sum_{j=0}^{k} \gamma_j, \quad (0 \leq k \leq n),$$

and assume that

$$\frac{1}{\rho} = \sum_{j=0}^{n} \gamma_j - 1 = \sigma_n = \mu_n - 1 > 0.$$

Following M.M. Dzhrbashchjan (see [6]), we consider the integro-differential operators

$$D^{(\sigma_0)} f(x) \equiv \frac{d^{-(1-\gamma_0)}}{dx^{-(1-\gamma_0)}} f(x),$$

$$D^{(\sigma_1)} f(x) \equiv \frac{d^{-(1-\gamma_1)}}{dx^{-(1-\gamma_1)}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x),$$

$$D^{(\sigma_2)} f(x) \equiv \frac{d^{-(1-\gamma_2)}}{dx^{-(1-\gamma_2)}} \frac{d^{\gamma_1}}{dx^{\gamma_1}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x),$$

$$\cdots\cdots\cdots\cdots$$

$$D^{(\sigma_n)} f(x) \equiv \frac{d^{-(1-\gamma_n)}}{dx^{-(1-\gamma_n)}} \frac{d^{\gamma_{n-1}}}{dx^{\gamma_{n-1}}} \cdots \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x).$$

Here we note that if $\gamma_0 = \gamma_1 = \cdots = \gamma_n = 1$, then obviously

$$D^{(\sigma_k)} f(x) = f^{(k)}(x), \quad (k = 0, 1, 2, \cdots, n).$$

The regional problems for the following equations are our object:

$$D^{(\sigma_n)} u - [\lambda + q(x)]\, u = 0, \quad 0 < \sigma_n < \infty, \tag{1}$$

$$u'' + D_{0x}^{\alpha} u + q(x) u = \lambda u, \quad 0 < \alpha < 1. \tag{2}$$

We shall consider various variants of equation (1). For

$$\gamma_0 = \gamma_1 = 1, \gamma_2 = \gamma_3 = \cdots = \gamma_{n-1} = 0,$$

equation (1) is transformed into

$$\frac{1}{\Gamma(1-\gamma_n)} \int_0^x \frac{u^{''}(t)}{(x-t)^{\gamma_n}} dt - [\lambda + q(x)] u(x) = 0, \tag{3}$$

which is called the fractional oscillating equation [5], and the operator $D^{(\sigma_2)}$ is called the operator of fractional differentiation in Caputo sense [5]. For

$$\gamma_0 = 1, \gamma_2 = \gamma_3 = \cdots = \gamma_{n-1} = 0, \gamma_n = 1,$$

equation (1) is transformed into

$$\frac{1}{\Gamma(1-\gamma_1)} \frac{d}{dx} \int_0^x \frac{u'(t)}{(x-t)^{\gamma_1}} dt - [\lambda + q(x)]\, u(x) = 0. \tag{4}$$

Equation (4) has been investigated as the modeling equation of fractional order $1 < \sigma < 2$ (see [3] and references therein). Further, if $\gamma_0 = \gamma_2 = \cdots = \gamma_n = 1$, then equation (1) will be written as

$$D^{(\sigma_n)} u = \frac{1}{\Gamma(1-\gamma_1)} \frac{d^{n-1}}{dx^{n-1}} \int_0^x \frac{d(t)}{(x-t)^{\gamma_1}} dt - [\lambda + q(x)]\, u(x) = 0. \tag{1$'$}$$

Point-to-point regional problem of Dirichlet $u(0) = 0$, $u(1) = 0$ for the fractional oscillatory equation was studied by one of the authors of this paper in [5]. There for the first time, Green's function for this kind of problems was constructed. In particular it was proved that at $q(x) = 0$, the problem is equivalent to the equation

$$u(x) = \frac{\lambda}{\Gamma(1-\gamma_2)} \left[\int_0^x (x-t)^{1-\gamma_2} u(t)dt - \int_0^1 x^{1-\gamma_2} (1-t)^{1-\gamma_2} u(t)dt \right].$$

The same problem for the modeling fractional differential equation of order $1 < \sigma < 2$ is equivalent to (see [3])

$$u(x) = \frac{\lambda}{\Gamma(1+\gamma_1)} \left[\int_0^x (x-t)^{\gamma_1} u(t)dt - \int_0^1 x(1-t)^{\gamma_1} u(t)dt \right].$$

The operator A, reversible to the operator B, induced by differential expression (1$'$) and natural regional conditions

$$u(0) = 0; D^{(\sigma_1)} u|_{x=0} = 0, \cdots, D^{(\sigma_{n-2})} u|_{x=0} = 0, u(1) = 0$$

looks like (see [3] and [6])

$$Au = \frac{1}{\Gamma(\rho^{-1})} \left[\int_0^x (x-t)^{\frac{1}{\rho}-1} u(t)dt - \int_0^1 x^{\frac{1}{\rho}-1} (1-t)^{\frac{1}{\rho}-1} u(t)dt \right].$$

Thus we shall note that if $\gamma_0 = \gamma_1 = \cdots = \gamma_n = 1$, then operator B will look like

$$Bu = \begin{cases} u^{(n)}, \\ u(0) = 0, u'(0) = 0, \cdots, u^{(n-2)}(0) = 0, u(1) = 0, \end{cases}$$

and Green's function $H(x, s)$ of the corresponding reverse operator will be rewritten as

$$H(x, s) = \begin{cases} \dfrac{(1-s)^{n-1}x^{n-1} - (x-s)^{n-1}}{(n-1)!}, & 0 \le x \le s \le 1, \\ \dfrac{(1-s)^{n-1}x^{n-1}}{(n-1)!}. \end{cases}$$

It was also stated in [3] that

$$Au = \int_0^x (x-t)u(t)dt - \int_0^1 x(1-t)u(t)dt.$$

In a certain sense, the disturbance of the operator can be expressed as

$$A_\varepsilon u = \frac{1}{\Gamma(2+\varepsilon)} \left[\int_0^x (x-t)^{1+\varepsilon}u(t)dt - \int_0^1 x(1-t)^{1+\varepsilon}u(t)dt \right].$$

It has been proved that eigenvalues of the operator A_ε are simple.

In [3] the expression for disturbance of the operator A has not been given explicitly. In the present work, the corresponding formula is obtained and it seems to the authors that this work pushes to include the theory of the differential equations of fractional order in the general frame of the theory of disturbance.

2. Expression and properties of Green's function

Let's consider the operator

$$A_\rho u = \frac{1}{\Gamma(\rho^{-1})} \left[\int_0^x (x-t)^{\frac{1}{\rho}-1}u(t)dt - \int_0^1 x^{\frac{1}{\rho}-1}(1-t)^{\frac{1}{\rho}-1}u(t)dt \right], \ 0 < \rho < 1/2$$

in $L_2(0, 1)$. It is known in [1] that the number λ is an eigenvalue of the operator A_ρ only if λ_0^{-1} is a zero point of the function $E_\rho\left(\lambda; \rho^{-1}\right)$ and the corresponding functions look like $\varphi_n^\rho(x) = x^{\frac{1}{\rho}-1}E_\rho\left(\lambda_n^\rho x^{1/\rho}; \rho^{-1}\right)$. Let λ_n^ρ be the n-th eigenvalue of the operator A_ρ, U_ρ be a limited area with straightened border dU_ρ such that $\lambda_n^\rho \in U_\rho$ and $[\sigma(A_\rho)\lambda_n^\rho] \cap \bar{U}_\rho = \emptyset$.

Let

$$P_{\lambda_n^\rho}(A_\rho) = -\frac{1}{2\pi i}\int\limits_{dU_\rho} R_\lambda(A_\rho)d\lambda.$$

The project of Riss for operator A_ρ corresponds to eigennumber λ_n^ρ.

Theorem 2.1. The projector $P_{\lambda_n^\rho}$ continuously depends on the parameter ρ.

Proof. The operators A_ρ are the sums of special one-dimensional operators and fractional-integration operators. It is known that the fractional-integration operators form in $L_P(0,1)$ $(P \geq 1)$ is, a half-group, continuous in uniform topology for all $\alpha > 0$ and strongly continuous for all $\alpha = 0$. Since A_ρ is continuous in uniform (operational) topology (i.e., $\| A_\rho - A_{\rho_0} \| \to 0$ for $\rho \to \rho_0$), $\| P_{\lambda_n^\rho} - P_{\lambda_n^{\rho_0}} \| \to 0$, when $\rho \to \rho_0$. This proves the theorem.

Corollary 2.1. Let $0 < \rho < \frac{1}{2}$, then

$$dim P_{\lambda_n^\rho}(A_\rho)L_2(0,1) = 1.$$

Proof. Since

$$\lim_{\rho\to\rho_0} \| A_\rho - A_{\rho_0} \| = 0,$$

for ρ and ρ_0 close enough,

$$\| P_{\lambda_n^\rho}(A_\rho) - P_{\lambda_n^{\rho_0}}(A_{\rho_0}) \| < 1.$$

Therefore according to Theorem 2.1, it follows that the spaces $P_{\lambda_n^\rho}(A_\rho)L_2(0,1)$ and $P_{\lambda_n^{\rho_0}}(A_\rho)L_2(0,1)$ have the same dimension. Since $dim P_{\lambda_n^{1/2}}(A_{1/2})L_2(0,1) = 1$, the function $dim P_{\lambda_n^\rho}(A_\rho)L_2(0,1) = 1$ for all $\rho \in (0,1)$. It follows from Corollary 2.1 that the eigennumbers of the operator A_ρ are simple, so are all the zero points of the function $E_\rho(\lambda; \rho^{-1})$. By the way, for the case $1/2 < \rho < 2$ this result was already stated in [3].

3. The basic oscillatory properties of operator A, ($0 < \rho < 1/2$)

Theorem 3.1. We have an expression

$$A_\varepsilon u = A + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots + \varepsilon^n A_n + \cdots, \quad \varepsilon > 0 \tag{1}$$

where

$$Au = \int\limits_0^x (x-t)u(t)dt - \int\limits_0^1 x(1-t)u(t)dt,$$

$$A_n u = \frac{1}{n!}\left[\int\limits_0^x (x-t)\ln^n(x-t)dt - \int\limits_0^1 x(1-t)\ln^n(x-xt)dt\right]$$

are operators with special kernels.

Proof: Let's rewrite the operator as

$$A_\varepsilon u = M_\varepsilon u + N_\varepsilon u$$

where

$$M_\varepsilon u = \int\limits_0^x K_\varepsilon(x,t)u(t)dt, \tag{2}$$

$$K_\varepsilon(x,t) = \begin{cases} (x-t)^{1+\varepsilon}, \quad t < x \\ 0, \quad t \geq x \end{cases} \tag{3}$$

and

$$N_\varepsilon = \int\limits_0^1 \widetilde{K}_\varepsilon(x;t)u(t)dt,$$

$$\widetilde{K}_\varepsilon(x,t) = \begin{cases} x^{1+\varepsilon}(1-t)^{1+\varepsilon}, \ t \neq 1 \\ 0, \ t = 1 \end{cases} \tag{4}$$

Considering the disturbance of the operator A_ε, we write

$$(A - A_\varepsilon)u = (M - M_\varepsilon)u - (N - N_\varepsilon)u.$$

First, we deal with $(M - M_\varepsilon)u$. Clearly,

$$(M - M_\varepsilon)u = \int\limits_0^1 [K(x,t) - K_\varepsilon(x,t)]u(t)dt.$$

Since

$$\begin{aligned} K(x,t) - K_\varepsilon(x,t) \quad &= \begin{cases} (x-t)[1-(x-t)^\varepsilon], \ t < x \\ 0, \ t \geq x \end{cases} \\ &= \begin{cases} (x-t)\left[\varepsilon\frac{\ln(x-t)}{1!} + \varepsilon^2\frac{\ln^2(x-t)}{2!} + \cdots + \varepsilon^n\frac{\ln^n(x-t)}{n!} + \cdots\right], \ t < x \\ 0, \ t \geq x, \end{cases} \end{aligned} \tag{5}$$

we have

$$(M - M_\varepsilon)u = \varepsilon\int\limits_0^1 K_1(x,t)u(t)dt + \cdots + \varepsilon^n\int\limits_0^1 K_n(x,t)u(t)dt + \cdots$$

where

$$K_n(x,t) = \begin{cases} \frac{(x-t)\ln^n(x-t)}{n!}, \ t < x \\ 0, \ t \geq x. \end{cases}$$

Thus

$$M_\varepsilon u = \int\limits_0^1 K(x,t)u(t)dt - \varepsilon \int\limits_0^1 K_1(x,t)u(t)dt - \cdots - \varepsilon^n \int\limits_0^1 K_n(x,t)u(t)dt - \cdots$$

Similarly,

$$N_\varepsilon u = \int\limits_0^1 \widetilde{K}(x,t)u(t)dt - \varepsilon \int\limits_0^1 \widetilde{K}_1(x,t)u(t)dt - \cdots - \varepsilon^n \int\limits_0^1 \widetilde{K}_n(x,t)u(t)dt + \cdots$$

where

$$\widetilde{K}_n(x,t) = \begin{cases} \dfrac{x(1-t)\ln^n(x-xt)}{n!}, \ t < 1 \\ 0, \ t = 1. \end{cases}$$

this proves Theorem 3.1.

Theorem 3.2. All eigenvalues $\lambda_n(\varepsilon)$ of operator A_ε are real.

Proof. As

$$\lambda_n(\varepsilon) = \pi n^2 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \tag{6}$$

$$\varphi_n(\varepsilon) = \sin nx + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \cdots, \tag{7}$$

where

$$\lambda_n = \sum_{k=1}^{n} (A_k\varphi_{n-k}, \sin nx), \tag{8}$$

$$\varphi_n = R\sum_{k=1}^{n} (\lambda_k - A_k)\varphi_{n-k}. \tag{9}$$

Here R-resulted resolvent of operator, corresponding to eigenvalue πn^2. This resolvent is integral operator with kernel

$$S(x,y) = \left[-\frac{y}{n}\cos ny \sin nx + \frac{1-x}{n}\sin ny \cos nx + \frac{1}{2n^2}\sin ny \sin nx \right], \ y \leq x$$

(if $y > x$ in a zero part of this middle it is necessary to change places y and x).

Clearly, that R transforms unequivocally H_0(H_0 is orthogonal addition of function $\sin \pi nx$) in itself and cancels $\sin \pi nx$.

From (8) follows that

$$\lambda_1 = (A_1 \sin nx, \sin x),$$

as kernel of operator A_1 have a real value, then $\sin \lambda_1 = 0$. From (9) it follows that

$$\varphi_1 = R(nk^2 - A_1) \sin nx$$

and the kernels of operators R and A_1 have real values, then $Im\varphi_1 = 0$. So, it's consistently possible to establish that all λ_i are real. And ε is real, then $\lambda u(\varepsilon)$ is real too.

Theorem 3.3. For eigenvalues $\lambda_n(x)$ and eigenfunction $\varphi_n(\varepsilon)$ of operator A_ε there are estimations

$$|\lambda_n(\varepsilon) - \pi n^2| < \frac{\pi(2n-1)}{2},$$

$$|\varphi_n(\varepsilon) - \sin nx| < \frac{1}{2}.$$

Proof. From (8) and (9), in assumption that

$$\|A_n u\| \leq p^{n-1}\{u\|u\| + b\|A_0 u\|\},$$

$$m = \|A_0\|; \frac{1}{d} = \|R\|, |\varepsilon| < \frac{1}{c}$$

where $c = \max\left\{\frac{8(a+mb)}{d}, 8p + 4\frac{a+mb}{d}\right\}$, follows simple formulas [8]

$$|\lambda(\varepsilon) - \lambda_0 - \varepsilon\lambda_1 - \cdots - \varepsilon^n\lambda_n| \leq \frac{d}{2}(|\varepsilon|c)^{n+1}, \tag{10}$$

$$|\varphi(\varepsilon) - \varphi_0 - \varepsilon\varphi_1 - \cdots - \varepsilon^n\varphi_n| \leq \frac{1}{2}(|\varepsilon|c)^{n+1}. \tag{11}$$

Let's calculate values of parameters a, b, c, d, m. First we find m: $m = \|A\| = \sup A$, where $\sup A$ is spectral radius of operator A. As,

$$\sup A = \pi^{-1},$$

then $m = \pi^{-1}$. Further

$$d = dist(\pi n^2; \Sigma'')$$

(d is an isolating distance) here Σ'' is spectrum of operator A^{-1} with unique excluded point πn^2. Clearly that $d = \pi(2n-1)$. To find other parameters a, b, c, we shall get estimation of the norm of the operator A_n

$$\|A_n\varphi\|_{L(0,1)} \leq \int_0^1 \int_0^1 |K_n(x,t)| \cdot |\varphi(t)| dt dt$$

here, $K_n(x,t)$ is kernel of the operator A_n

$$\int_0^1\int_0^1 |K_n(x,t)|\cdot|\varphi(t)|dtdx = \int_0^1\int_0^x \left|\frac{(x-t)\ln^n(x-t)}{n!}\right| |\varphi(t)|dtdx$$

$$= \int_0^1 |\varphi(t)| \int_0^{1-t} \frac{z|\ln^n z|}{n!}dzdt \le \frac{1}{n!}\int_0^1 z|\ln^n z|dz \cdot \|\varphi\|_{L_1(0,1)}.$$

Let's calculate integral $\int z\ln z^n dz$.

$$\int z\ln z^n dz = \frac{z^2(\ln z)^n}{2} - \frac{nx^2(\ln x)^{n-1}}{2^2} + \cdots + \frac{(-1)^{n-1}n(n-1)(n-2)\cdots 2}{2^{n-1}}(\frac{x^2}{2} - \frac{1}{2^2}).$$

From here $\|A_n\| \le \dfrac{1}{2^{n+1}}$. Now, we will take $a = \frac{1}{4}$, $p = \frac{1}{2}$, and $b = 0$. Since

$$c = \max\left\{8\frac{a+mb}{d}, 8p + 4\frac{a+mb}{d}\right\},$$

we have

$$c = \max\left\{\frac{2}{d}, 4 + \frac{1}{d}\right\} = 4 + \frac{1}{d} = 5.$$

Then

$$|\lambda(\varepsilon) - \lambda| \le \frac{1}{2}\frac{\pi(2n-1)}{2}.$$

Precisely we found

$$|\varphi_\varepsilon - \varphi_0| \le \frac{1}{2},$$

and prove Theorem 3.3.

Theorem 3.4. Let $u_0(x)$, $u_1(x)$, $\cdots$, $u_n(x)$, $\cdots$ be eigenfunctions of operator A_ρ, and λ_0, λ_1, $\cdots$, λ_n, $\cdots$ be corresponding eigennumbers, then own fluctuation with the least frequency $\varphi_0(x)$ has no units, i.e., $u_0(x) \ne 0$, $0 < x < 1$.

Proof. Let λ_0 be the least eigenvalue of operator A_ρ. Then

$$u_0(x) = E_\rho(-\lambda_0 x; \rho^{-1}) = \sum_{k=0}^{\infty} \frac{(-\lambda_0 x)^k}{\Gamma(\rho^{-1} + \rho_k^{-1})}.$$

Assume that function $u_0(x)$ has no zero in $(0,1)$. Let, the function $u_0(x)$ in a point $x_0 \in (0,1)$ equal to zero, i.e.

$$u_0(x_0) = E_\rho(-\lambda_0 x_0; \rho^{-1}) = \sum_{k=0}^{\infty} \frac{(-\lambda_0 x_0)^k}{\Gamma(\rho^{-1} + \rho_k^{-1})} = 0.$$

That's to say, the number $-\lambda_0 x_0$ is a zero point of $E_\rho(z, \rho^{-1})$. However, $-\lambda_0 x_0 < \lambda_0$ as $x_0 \in (0,1)$, and we assume that the least zero is λ_0. Then the received contradiction proves the Theorem 3.4.

Remark 3.1. It is possible to show also precisely that $u_1(x)$ in an interval has exactly one unit, etc.

Theorem of existence of the basis is made of root operators $A_\rho, (0 < \rho < \frac{1}{2})$. Problem of completeness of systems of eigenfunctions of the operator induced by differential expression

$$l(D)_n = \begin{cases} \frac{1}{\Gamma(1-\gamma_1)} \frac{d^{n-1}}{dx^{n-1}} \int\limits_0^x \frac{u'(t)}{(x-t)^{\gamma_1}} dt - (\lambda + q(x))u \\ u(0) = 0, D^{\sigma_1} u|_{x=0} = 0, \cdots, D^{\sigma_{n-2}} u|_{x=0} = 0, u(1) = 0. \end{cases}$$

When $q(x)$ is half-limited function, the case is studied(see [3] and references therein). If completeness of system of eigenfunctions that a following question is proved is formulated so: whether It is possible to make basis of eigenfunctions of this operator.

Lemma 3.1. *The operator A is dissipative.*

Proof: we shall consider the operator

$$Au = \int\limits_0^x (x-t)^{1+\varepsilon} u(t) dt - \int\limits_0^1 x^{1+\varepsilon} (1-t)^{1+\varepsilon} u(t) dt.$$

Let's introduce a designation

$$v(t) = \int\limits_0^x (x-t)^{1+\varepsilon} u(t) dt - \int\limits_0^1 x^{1+\varepsilon} (1-t)^{1+\varepsilon} u(t) dt.$$

From this expression follows that

$$u(x) = Dv \in L_2(0,1).$$

Let's consider all over again product

$$
\begin{aligned}
&(Au,\overline{u})\left[\int\limits_0^x (x-t)^{1+\varepsilon}u(t)dt-\int\limits_0^1 x^{1+\varepsilon}(1-t)^{1+\varepsilon}u(t)dt\right]\overline{u}(x)\\
&=\int\limits_0^1\left[\int\limits_0^x (x-t)^{1+\varepsilon}u(t)dt-\int x^{1+\varepsilon}(1-t)^{1+\varepsilon}u(t)dt\right]\overline{u}(x)dx\\
&=\int\limits_0^1 v(x)D_x\overline{v}dt=\int\limits_0^1 v(x)\tfrac{d}{dx}\left(\int\limits_0^x \tfrac{\overline{v'}(t)}{(x-t)^{\varepsilon}}dt\right)dx\\
&=\int\limits_0^1 v(x)\left(\int\limits_0^x \tfrac{\overline{v}(t)}{(x-t)^{\varepsilon}}dt\right)dx\\
&=v(x)\int\limits_0^1 v(x)\left(\int\limits_0^x \tfrac{v'(t)}{(x-t)^{\varepsilon}}dt\right)\Bigg|_0^1-\int\limits_0^1\left(\int\limits_0^x \tfrac{\overline{v'}(t)}{(x-t)^{\varepsilon}}dt\right)\overline{v'(x)}dx\\
&=-\int\limits_0^1\left(\int\limits_0^x \tfrac{\overline{v'}(t)}{(x-t)^{\varepsilon}}dt\right)\overline{v'(t)}dx,
\end{aligned}
$$

as $v(0)=v(1)=0$. Let's define $v'(x)=z(x)$, then $(Au,\overline{u})=-(J^{\varepsilon}z,\overline{z})$. Now, by virtue theorem of Matsaev-Polant, it follows that values of form $(J^{\varepsilon}z,\overline{z})$ lays in angle $|argz|<\frac{\pi\varepsilon}{2}$, that's proves Lemma 3.1. In papers [1] and [23], a dissipativity of operator $l(D)$ is proved.

4. Questions basis systems of eigenfunctions.

Theorem 4.1. The system of eigenfunctions of operator A forms basis of the closed linear hull.

Proof. Let's remind that system of vectors $\{u_1,u_2,\cdots,u_n,\cdots\}$ forms basis of the closed linear hull $G\subset H$, if executed inequality

$$
m\sum_{k=1}^{n}|c_k|^2\leq\|\sum_{k=1}^{n}c_k\varphi^2\|^2\leq M\sum_{k=1}^{n}|c_k|^2
$$

where m, M are positive constants independent of values c_1, c_2, c_3, $\cdots$, c_n $(n=1,2,3,\cdots)$. Any vector $f(f\in G)$ in that case unequivocally reveals in a series $f=\sum\limits_{n=1}^{\infty}c_ku_k$, $(\sum\limits_{n=1}^{\infty}|c_k|^2<\infty)$. It is known that if the spectrum of dissipative operator with quite continuous imaginary component "is pressed enough" to a real axis, then it is possible to make basis of their linear closed hull of eigenvectors of this operator. We need theorem of Glazman.

Theorem of Glazman: Let A be the linear limited dissipative operator with quite continuous imaginary component, possessing infinite system of eigenvectors $\{u_k\}|_{k=1}^{\infty}$, normalized by condition $(u_k,u_u)=1(k=1,2,3,\cdots)$, and $\{\lambda_k\}|_{k=1}^{\infty}$ be corresponding sequence of different eigenvalues. If the condition

$$
\sum\frac{Im\lambda_j Im\lambda_k}{|\lambda_j-\overline{\lambda}_k|}<\infty
$$

is satisfied, then the system $\{\varphi_k\}|_{k=1}^{\infty}$ is a Bari-Riss's basis of closed linear area. Since for $0 < \rho < 1/2$ all eigennumbers of operator A_ρ are real, due to theorem of Glazman, proof of Theorem 4.1 is follows. Certainly, precisely also it is possible to prove similar statements and for more general problems.

5. Partial problem of eigenvalues of the operator $A_\rho, 0 < \rho < 1/2$

In [1] and [3], there have been allocated areas in a complex plane where there are no eigenvalues of the operator A_ρ for any ρ. Here we, as well as above, shall assume that $0 < \rho < 1/2$. In applied problems the greatest interest represents usually definition of first eigenvalues, therefore we shall solve this problem for eigenvalues of the operator A_ρ.

Let's consider the operator

$$A = \frac{1}{\Gamma(\rho^{-1})}\left[\int\limits_0^x (x-t)^{1/\rho}u(t)dt - \int\limits_0^1 x^{1/\rho-1}(1-t)^{1/\rho-1}u(t)dt\right] = A_0u + A_1u.$$

As $0 < \rho < 1/\rho$, then operator A_0 and A_1 are reverse. Therefore $sp(A) = spA_0 + spA_1$.

Clearly that

$$spA_0 = 0, \quad spA_1 = \frac{1}{\Gamma(2-1/\rho)}$$

Therefore it is meaningful a determinant

$$D_A(\lambda) = \sum_{j=1}^{\infty}(1-\lambda\mu_j), \quad \mu_k = \frac{1}{\lambda_k}.$$

Earlier it has been established that number λ is eigenvalue of operator A in only case when λ^{-1} is zero of function $E_\rho(\lambda;\rho^{-1})$. Because whole functions $D_a(A)$ and $E_\rho(\lambda;\rho^{-1})$ are of a zero kind, then we shall notice that

$$D_A(\lambda) = cE_\rho(\lambda;2)$$

where c - is a while unknown constant. Let's consider a logarithmic derivative

$$[lnD_A(\lambda]' = \frac{D'_A(\lambda)}{D_A(\lambda)} = -\sum_{j=1}^{\infty}\frac{\lambda_j}{1-\lambda\lambda_j} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(\lambda_j^k k)\lambda^{k-1}, \quad (|\lambda| < \lambda_1^{-1})$$

or

$$[\ln D_A(\lambda)]' = \frac{D'_A(\lambda)}{D_A(\lambda)} = -sp\left(A(J-\lambda A)^{-1}\right) = -\sum \chi_{n+1}\lambda^n.$$

where $\chi_n = spA^n$.

Let $D_A(A) = \sum\limits_{0=1}^{\infty} a_k\lambda^k$ be a representation by Taylor's series of the function $D_A(A)$. Let's establish interrelation between χ_k and a_k. As

$$D_A(\lambda) = E_\rho(\lambda; \rho^{-1}) = \sum \frac{\lambda^k}{\Gamma(\rho^{-1} + k\rho^{-1})}$$

then $a_n = \frac{1}{\Gamma(\rho^{-1}+k\rho^{-1})}$. Further

$$\frac{D'_A(\lambda)}{D_A(\lambda)} = \frac{\sum(na_n\lambda^n)}{\sum a_n\lambda_n} = -\sum \chi_{n+1}\lambda^n,$$

we obtain a recurrent parity

$$\chi_{n+1} + \sum a_s\chi_{n+1-s} = -(n+1)a_{n+1},$$

$$\chi_{n+1} = -\sum a_{n-k}\chi_k - na_n = \left[-\sum \frac{(-1)^{n-k}}{\Gamma(2-\beta(n-k))}\chi_k - n\frac{(-1)^n}{\Gamma(2-n\beta)}\right];$$

$$\chi_1 = \frac{1}{\Gamma(2+1/\rho)}; \quad 2a_2 + a_1\chi_1 = -\chi_2; \chi_2 = a_1^2 - 2a_2;$$

$$-\chi_3 = \sum_{s=1}^{2} 3a_3^2\chi_{3-s}; \quad \chi_3 = a_2a_1 + a_1(2a_2 - a_1^2) - 3a_2$$

As

$$\frac{1}{\chi_1} < \lambda_1 < \frac{\chi_1}{\chi_2},$$

then

$$\frac{1}{\Gamma(2+1/\rho)} < \lambda_1 < \frac{2}{\Gamma(2+1/\rho)}.$$

Let we have set $\{\gamma_0, \gamma_1, \gamma_2\}$ of three numbers $0 \le \gamma_j \ge 1, (j = 0, 1, 2)$. Following Dzhrbashjan M.M. [9], we shall designate

$$\sigma_k = \sum_{j=0}^{k} \gamma_j - 1, \mu_k = \sigma_k + 1 = \sum_{j=0}^{k} \gamma_k, (k = 0, 1, 2).$$

Let's assume that

$$\frac{1}{\rho} = \sum_{j=0}^{2} \gamma_j - 1 = T_2 = \mu_2 - 1 > 0.$$

As $1 < \sigma_2 < 2$, then $1 < \frac{1}{\lambda} < 2$, i.e., $1/2 < \rho < 1$. Last assumption is very important.

Let's enter into consideration differential operators [5]

$$\widetilde{\mu}_k = \widetilde{\sigma}_k + 1 = \sum_{j=0}^{k} \gamma_{2-j}, \quad (k = 0, 1, 2),$$

$$D_1^{(\widetilde{\sigma_0})} f(x) \equiv \frac{d^{-(1-\gamma_2)}}{d(1-x)^{-(1-\gamma_2)}} f(x),$$

$$D_1^{(\widetilde{\sigma_1})} f(x) \equiv \ -\frac{d^{-(1-\gamma_1)}}{d(1-x)^{-(1-\gamma_1)}} \frac{d^{\gamma_2}}{d(1-x)^{\gamma_2}} f(x),$$

$$D_1^{(\widetilde{\sigma_2})} f(x) \equiv \frac{d^{-(1-\gamma_0)}}{d(1-x)^{-(1-\gamma_0)}} \frac{d^{\gamma_1}}{d(1-x)^{\gamma_1}} \frac{d^{\gamma_2}}{d(1-\gamma)^{\gamma_2}} f(x).$$

And now, problem $\widetilde{A}$ may be put as follows. In class $L_2(0,1)$ (or $L_1(0,1)$) we find nontrivial solution of the equation

$$D^{(\widetilde{\sigma_2})} z - \{\lambda + q(x)\} z = 0, x \in [0,1),$$

satisfying the boundary conditions

$$D_1^{(\widetilde{\sigma_0})} z \Big|_{x=0} \cos\alpha + D_1^{(\widetilde{\sigma_1})} z \Big|_{x=0} \sin\alpha = 0,$$

$$D_1^{(\widetilde{\sigma_0})} z \Big|_{x=0} \cos\beta + D_1^{(\widetilde{\sigma_1})} z \Big|_{x=0} \sin\beta = 0.$$

the associated problem gives essentially, new results, in the case when the order of the fractional differential equation is less than one. To this case we shall devote separate paper. To show how to transfer the obtained results on a case of the differential equations of order higher than two, we consider the following problem.

6. Operators of transformation.

V.A. Marchenko [7] at the solution of a reverse problem for equation

$$y'' - q(x)y + \lambda y = 0, \quad (0 \le x \le 1) \tag{1}$$

builds the operator of transformation, translating a solution of equation (1) in a solution of the equation

$$y'' + \lambda y = 0. \tag{2}$$

By means of Green's function, with initial data $y(0) = 0$, (2) corresponds to the differential operator

$$l(y) = y'' - q(x)y,$$

and the integral operator

$$Ay = \int_0^x G(x,\xi)y(\xi)d\xi \quad (0 \le x \le 1),$$

where

$$G(x,x) = 0, \; \frac{d}{dx}G(x,\xi)\Big|_{\xi=x} = 1.$$

From Marchenko's results follows, that operator A is linearly equivalent to the operator of repeated integration [9]

$$J^2 y = \int_0^x (x-\xi)y(\xi)d\xi$$

with $y(\xi) \in L^2[0,1]$.

In given paper, similar results we shall obtain for the differential equations of the fractional order $\sigma : (1 < \sigma \le 2)$. In the further we shall use operators of fractional differentiation of Dzhrbashjan M.M. [9], which are defined as follows.

Now, let's consider a problem of Cauchy type

$$J\{y;\gamma_0,\gamma_1,\gamma_2\} = D^{\sigma_2}y - \{\lambda + q(x)\}y = 0, \quad x \in (0,1], \tag{3}$$

$$\frac{d}{dx}D^{\sigma_0}\Big|_{x=0} = C_0, \; \frac{d}{dx}D^{\sigma_1}\Big|_{x=0} = C_1, \tag{4}$$

with $q(x) \in C[0,1]$.

Let's note that some results of this paper earlier have been published in [1] and [7]. In these works, operator of transformation translating a solution of the equation (3) in a solution of the equation $D^{\sigma_2}y - \lambda y = 0$ has been constructed. Let $y(x,\lambda)$ be a solution of a problem of Cauchy type. Then it is known [2] that the identity takes place

$$y(x;\lambda) = C_0x^{\mu_0-1}E_\rho(\lambda x^{1/\rho};\mu_0) + C_1x^{\mu_1-1}E_\rho(\lambda x^{1/\rho};\mu_1) +$$
$$\int_0^x (x-\tau)^{1/\rho}E_\rho(\lambda(x-\tau)^{1/\rho};\frac{1}{\rho})q(\tau)y(\tau;\lambda)d\tau, \tag{5}$$

where

$$E_\rho(x,\mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + \frac{k}{\rho})} \tag{6}$$

is a function of a Mittag-Leffler's type.

In the further we shall consider the case that

$$C_0 = 0, C_1 = 1, \lambda = iz, Imz = 0, \mu_1 = 1, |q(x)| < 1$$

. As $\mu_1 = 1$, then

$$|E_\rho(\lambda x^{1/\rho}; \mu_1)| \leq Cx^{-1/\rho}, (1 \leq x < \infty)$$

(see [5]). In [7], as

$$y(x;\lambda) = x^{\mu_1 - 1} E_\rho(\lambda x^{1/\rho}; \mu_1) + \int\limits_0^x (x-\tau)^{1/\rho - 1} E_\rho(\lambda(x-\tau)^{1/\rho}; \frac{1}{\rho}) q(\tau) y(\tau;\lambda) d\tau, \quad (7)$$

that of general theory Volterra follows that the equation (7) can be solved by a method of iterations. As function $q(x)$ is continuous, then we have

$$y(x,\lambda) = E_\rho(\lambda x^{1/\rho}; \mu_1) + \varphi_1(\lambda, x) + \varphi_2(\lambda, x) + \cdots, \quad (8)$$

where

$$\begin{aligned} \varphi_n(\lambda, x) = \quad & \int\limits_0^x d\tau_1 \cdots \int\limits_0^{\tau_n - 1} d\tau_1 \cdots q(\tau_n) \\ & E_\rho\left(\lambda(x-\tau_1)^{1/\rho}; \mu_1\right) \cdots E_\rho\left(\lambda(\tau_{n-1} - \tau_n)^{1/\rho}; \mu_1\right) E_\rho\left(\lambda \tau_n^{1/\rho}; \mu_1\right). \end{aligned}$$

Obviously that a series (8) converges in regular intervals in each interval $(0; l), l < 1$.

Theorem 6.1. There is the operator of transformation V, transforming a solution $E_\rho(\lambda x^{1/\rho}; \mu_1)$ of equation (5) in a solution $y(x;\lambda)$ of equation (3).

Lemma 6.1. The parity

$$(J - \lambda J^{1/\rho})^{-1} \frac{1}{\Gamma(\rho^{-1})} = E_\rho(\lambda x^{1/\rho}; 1). \quad (9)$$

holds.

Proof. Let $f(x) \in L_1(0,1)$, for $\rho > 0$ and λ is any complex parameter, then the integrated equation

$$u(x) = f(x) + \frac{\lambda}{\Gamma(\rho^{-1})} \int\limits_0^x (x-1)^{1/\rho} u(t) dt \quad (10)$$

has an unique solution

$$u(x) = f(x) + \lambda \int\limits_0^x (x-t)^{1/\rho - 1} E_\rho\left(\lambda(x-t)^{1/\rho}; \frac{x}{\rho}\right) f(t) dt, \quad (11)$$

belonging to a class $t_1(0,1)$.

In (10), let $f(x) = \frac{1}{\Gamma(\rho^{-1})}$, we obtain

$$u(x) = \frac{1}{\Gamma(\rho^{-1})} + \frac{1}{\Gamma(\rho^{-1})} \int_0^x (x-t)u(t)dt. \tag{12}$$

Due to (11) and (12), we have

$$u(x) = \frac{1}{\Gamma(\rho^{-1})} + \frac{1}{\Gamma(\rho^{-1})} \int_0^x (x-t)^{1/\rho-1} E_\rho \left(\lambda(x-t)^{1/\rho}; \frac{x}{\rho}\right) dt. \tag{13}$$

Using the known formula of Dzhrbashjan M.M.

$$\frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} E_\rho \left(\lambda t^{1/\rho}; \mu\right) t^{\mu-1} dt = z^{\mu+\alpha-1} E_\rho \left(\lambda z^{1/\rho}; \mu+\alpha\right), \ (\mu > 0, \alpha > 0),$$

let's calculate the integral

$$\frac{1}{\Gamma(\rho^{-1})} \int_0^x (x-t)^{1/\rho-1} E_\rho \left(\lambda(x-t)^{1/\rho}; \frac{x}{\rho}\right) dt = \lambda x^{1/\rho} E_\rho \left(\lambda x^{1/\rho}; 1 + \frac{1}{\rho}\right). \tag{14}$$

From (13) and (14) it follows that

$$u(x) = \frac{1}{\Gamma(\rho^{-1})} + \lambda x^{1/\rho} E_\rho \left(\lambda x^{1/\rho}; 1 + \frac{1}{\rho}\right) = E_\rho \left(\lambda x^{1/\rho}; 1\right).$$

Thus the lemma 6.1 is proved.

Lemma 6.2. Let $y(x,\lambda)$ be a solution of a problem of Cauchy type (3)-(4), then

$$y(x,\lambda) = (J+\lambda)KA^{-1}f,$$

where

$$Au = \int_0^x (x-t)^{1/\rho-1} u(t)dt, \quad K = (J + Aq(x)), \quad f = 1.$$

Proof. Obviously that a solution of the equation

$$\begin{cases} D^\sigma y + q(x)y - \lambda y = 0, \\ D^{\sigma_0} y|_{x=0} = 0, \\ D^{\sigma_1} y|_{x=0} = -1 \end{cases}$$

coincides with a solution of the equation

$$y + A^{-1}q(x)y + \lambda Ay = f.$$

Let's designate $K = (J + A^{-1}q(x))$, then we shall obtain

$$y = (J + \lambda xA^{-1})^{-1}f,$$

which proves Lemma 6.2.

Lemma 6.3. Operator B is linearly equivalent to operator $J^{1/\rho}$.

Proof. According to Theorem 6.1 and Lemma 6.2, it follows that

$$VE_\rho(\lambda x^{1/\rho}; 1) = (I + \lambda KA^{-1})^{-1}f. \tag{15}$$

and from (9) and (15),

$$(J + \lambda B)^{-1}f = V(J + J^{1/\rho})^{-1} + \frac{1}{\Gamma(\rho^{-1})}. \tag{16}$$

Dividing both parts of a parity (16) in series on degrees λ, we obtain

$$\lambda^0 f = \lambda^0 V\frac{1}{\Gamma(\rho^{-1})}, Kf = VJ^{1/\rho}\frac{1}{\Gamma(\rho^{-1})}.$$

From these parities follows that $K = VJ^{1/\rho}V^{-1}$, as proves Lemma 6.3.

Theorem 6.2. Operator B is a monocell Volterr's operator.

Proof. As the operator $J^{1/\rho}$ is monocell, then by virtue of lemma 6.3, the operator B is monocell too.

Theorem 6.2 can be used for solution of reverse problems for the differential equations of the fractional order.

Chapter 2. *The Storm-Loiuville Problems for a Second Order Ordinary Differential Equation with Fractional Derivatives in the Lower Terms.*

1. About some problems from the theory of the equations of the mixed type leading boundary problems for the differential equations of the second order with fractional derivatives.

Let's consider equation

$$\begin{cases} u'' + a_0(x)u' + \sum\limits_{i=1}^{m} a_i(x)D_{0x}^{\alpha_i}\omega_i(x)u + u_{m+1}(x)u = f(x), \\ u(0)\cos\alpha + u'(0)\sin\alpha = 0, \\ u(1)\cos\beta + u'(1)\sin\beta = 0. \end{cases} \tag{1}$$

where $0 < \alpha_m < \cdots < \alpha_1 < 1$, and D^α is a operator of fractional differentiation of order α

$$D^\sigma u = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int\limits_0^x \frac{u(t)}{(x-t)^\sigma}dt, \ 0 < \sigma < 1.$$

For the equations of a kind (1) it is equivalent many direct and return problems, associated with a degenerating hyperbolic equation and equation of the mixed hyperbolic-parabolic type are reduced. In [3] it is shown that to a problem (1) the analogue of problem of Tricomy type for the hyperbolic-parabolic equation with operator Gellerstendta in a body is reduced.

On Euclid planes with the cartesian orthogonal coordinates x and y we shall consider the modeling equation in private derivatives of the mixed (parabolic-hyperbolic) type

$$|y|^{mH(-1)}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^{1+H(-y)}u}{\partial y^{1+H(-y)}} \tag{2}$$

where $m = const > 0$, $H(y)$ is the function of Heavyside, and $u = u(x, y)$.

The equation (1) in top half-plane coincides with Fourier's equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$$

and in lower half-plane coincides with

$$(-y)^m\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} \tag{3}$$

which for $y = 0$ transforms to the equation of parabolic type.

Let Ω be an area limited by pieces of straight lines

$$AA_0 : x = 0,$$

$$A_0B_0 : y = y_0,$$

$$B_0B : x = r,$$

and characteristics

$$AC : x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0$$

and

$$BC : x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = r$$

of equation (3);

$$\Omega^+ = \{(x, y) : 0 < x < r, 0 < y < y_0\}$$

be parabolic of mixed area Ω; Ω^- be part of area Ω, laying on lower half-plane $y<0$ and limited by characteristics

$$AC, 0 \leq x \leq \frac{r}{2}, BC, \frac{r}{2} \leq x \leq r$$

and piece

$$AB :]0, r[= \{(x,0) : 0 < x < r\};$$

$$u = \begin{cases} u^+(x,y), & \forall (x,y) \in \Omega^+, \\ u^-(x,y), & \forall (x,y) \in \Omega^-. \end{cases} \tag{4}$$

where D_{0x}^{l} is operator of fractional integro-differention of Storm-Liouville of order $\mid l \mid$ and with beginning point 0.

Let's consider a problem of Tricomy type, for equation (1) in Ω, with nonlocal condition of linear conjugation in following statement.

Problem 1. Find the regular decision of the equation (1) in the areas Ω^+, Ω^-, with the following properties

$$u^+ \in C(\bar{\Omega}^+) \cap C^1(\Omega^+ \cup]0,r[), u^- \in C(\bar{\Omega}^-) \cap C^1(\Omega^- \cup]0,r[), \tag{5}$$

$$u^+(x,0) = u^-(x,0) - \lambda D_{0x}^{-2} u^-(t,0), \quad 0 \leq x \leq r; \tag{6}$$

$$\left. \frac{\partial(u^+ - u^-)}{\partial y} \right|_{y=0} = 0, \quad 0 < x < r; \tag{7}$$

$$u^+(0,y) = \varphi_0(y), u^-(r,y) = \varphi_r(y), \quad 0 \leq y \leq y_0; \tag{8}$$

$$u^-\big|_{AC} = \psi(0), 0 < x < r; \tag{9}$$

where $\bar{\Omega}$ is a closure of Ω, λ is a spectral parameter, $\varphi_0(y)$ and $\varphi_r(y)$ are given functions of class $C^1[0,r]$, and

$$\psi(x) = u\left[\frac{x}{2}, -(\frac{m+2}{4}x)^{2/(m+2)}\right]$$

is a given function of class $C^3[0,r]$.

Nonlocal condition of conjugation set by the equation (6), and local regional conditions (7),(8) coincide with Tricomy conditions. For $\lambda = 0$, the problem (1) identity to analogue of Tricomy type problem.

From (6) and (9) for $x = 0$ and (8), for y=0, follows that equality $\varphi_0(0) = \psi(0)$ is a necessary condition of the coordination of regional data.

We have a following lemma

Lemma 1.1. *Let (4) is a solution of Problem 1*

$$\tau(x) = u^-(x,0), \quad \nu(x) = \frac{\partial u^-}{dy}\bigg|_{y=0} = 0; \quad x^{-\beta}\nu(x) \in L[0,r]. \tag{10}$$

Then

$$\tau''(x) - \lambda\tau(x) = \nu(x), \tag{11}$$

$$\Gamma(2\beta)D_{0x}^{1-2\beta}\tau(t) = \Big[\beta_m\nu(x) + x^\beta D_{0x}^{1-\beta}\psi(t)\Big]\Gamma(\beta), \tag{12}$$

$$\tau(0) = \varphi_0(0), \tau(r) - \lambda\int\limits_0^r (r-t)\tau(t)dt = \varphi_r(0), \tag{13}$$

where $\Gamma(z)$ is a Euler gamma-function, and

$$\beta = \frac{m}{2m+4}, \ \beta_m = \frac{\Gamma(2-2\beta}{\Gamma(1-\beta)}(2-4\beta)^{2\beta-1}.$$

In fact, as $u = u^+(x,y)$ is the solution of problem (1) in area Ω^+, then from (2) by (10), we have $u^{+\prime\prime}(x,0) = \nu(x)$. On the other hand, according to nonlocal condition of conjunction (11),

$$u^{+\prime\prime}(x,0) = r''(x) - \lambda\tau(x) \tag{14}$$

. From this equality the parity (11) follows between $\tau(x)$ and $\nu(x)$. The equation (12) represents other form of record of the known equation to theories of the mixed type of a functional parity between $\tau(x)$ and $\nu(x)$, the parabolic expression introduced on a line from area Ω^- for Gellester's equation (3) is consequence (6) and (8).

Excepting 89 from system (11)-(12), we have

$$L_{\beta\tau(x)}(x) = \lambda\tau + \psi_\beta(x), \tag{15}$$

where

$$L_{\beta\tau(x)}(x) = \tau''(x) - \mu_\beta D_{0x}^{1-2\beta}\tau(t), \tag{16}$$

$$\mu_\beta = \frac{\Gamma(2\beta)}{\beta_m\Gamma(\beta)}, \psi_\beta = -\frac{1}{\beta_m}x^\beta D_{0x}^{1-\beta}\psi(t) \tag{17}$$

Hence, at observance of a condition of a lemma 1.1 function $\tau(x) = u^-(x,0)$ should be the decision of a following nonlocal problem.

Problem 2. Find the solution $\tau(x)$ of equation (11) of class $C^2[0,r]\cap C^1[0,r]$, satisfying to condition (9).

Passing to research of structural and qualitative properties of solution of Problem 2, we shall notice that any solution of the equation (11) of a class $C^2[0,r]\cap C^1[0,r]\cap]0,r[$ will be a solution of the equation

$$\tau(x)=\tau'(0)x+\tau(0)+\mu_\beta D_{0x}^{-2}D_{0t}^{1-2\beta}\tau(\xi)+\lambda D_{0x}^{-2}\tau(t)+D_{0t}^{1-2\beta}\psi_\beta(t). \tag{18}$$

Equation (15) reverse out from equality (11) after application to both its parts of the operator $D_{0t}^{1-2\beta}$.

It is easy to see that

$$D_{0x}^{-2}D_{0t}^{1-2\beta}\tau(\xi)=\int\limits_0^x(x-t)\frac{\partial}{\partial t}D_{0t}^{-2\beta}\tau(\xi)dt=D_{0x}^{-1}D_{0t}^{-2\beta}\tau(\xi)=D_{0x}^{-1}D_{0t}^{-2\beta}\tau(t).$$

In view of it from (15) by virtue of (9) we have

$$\varphi_r(0)-\varphi_0(0)=rr'(0)+\mu_\beta D_{0r}^{-1-2\beta}\tau(t)+D_{0r}^{-2}\psi_\beta(t). \tag{19}$$

Substituting value $r'(0)$ (16) in equation (15), we have

$$\tau(x)=\mu_\beta D_{0x}^{-\alpha}\tau(t)+a_1xD_{0x}^{-\alpha}\tau(t)+\lambda D_{0x}^{-2}\tau(t)+f_0(x). \tag{20}$$

where $a_1=\frac{-\mu_\beta}{r}, \alpha=1+2\beta=2(m+1)/(m+2)$,

$$f_\beta(x)=D_{0x}^{-2}\psi_\beta(t)+\varphi_0(0)+\frac{x}{r}[\varphi_r(0)-\varphi_0(0)-D_{0x}^{-2}\psi_\beta(t)]. \tag{21}$$

The equation (17) is integrated Fredholm's equation of the second order and it is equivalent to Problem 2.

2. Formulas for calculation of eigenvalues of one boundary problem.

Let's introduce some formulas from the theory of disturbance which we need in the further. Let $T(\chi,\varepsilon)$ be a linear operational bunch

$$T(\chi,\varepsilon)=T+\chi T'+\varepsilon T'',$$

where T is a complete self-adjoined operator, all of which eigenvalues isolated and have frequency rate equal to 1; T' and T'' are certain in the same Hilbert's space as T; T is limited.

$$T(\chi,\varepsilon)-\zeta=T-\zeta+\chi T'+\varepsilon T''=\left[J+(\chi T'+\varepsilon T'')R(\zeta,T)\right](T-\zeta),$$

if ζ is not eigenvalue of $T(\chi, \varepsilon)$. Then, a resolvent

$$R(\zeta, x, \varepsilon) = R(\zeta, T)\left[J - (\chi T' - \varepsilon T'')R(\zeta, T)\right]^{-1}$$

exists, if a term

$$[J - (\chi T' - \varepsilon T'')R(\zeta, T)]^{-1}$$

may be determine as a Niemann's series

$$[J - (\chi T' - \varepsilon T'')R(\zeta, T)]^{-1} = \sum_{n=0}^{\infty}[-(\chi T' - \varepsilon T'')R(\zeta, T)]^{n}$$

i.e. A operator, reverse to operator $\left[J + (\chi T' + \varepsilon T'')R(\zeta, T)\right]^{-1}$, exists and equals to

$$\sum[-(\chi T' + \varepsilon T'')R(\zeta, T)]^{n},$$

and for this a sufficient condition is

$$\|\chi T' + \varepsilon T''\| \leq \|R(\zeta, T)\|. \tag{1}$$

So, if the parity (1) is fair, then

$$R(\zeta, T)\left[J + (\chi T' + \varepsilon T'')R(\zeta, T)\right]^{-1} = R(\zeta, T)\sum_{p=0}^{\infty}[-(\chi T' + \varepsilon T'')R(\zeta, T)]^{p} = R(\zeta, \chi, \varepsilon). \tag{2}$$

Let λ be a eigenvalue of operator $T = T(0,0)$ of frequency rate $m = 1$, Γ is closed positively focused contour containing on resolvent set and containing only eigenvalues λ of T. Let's consider the projector

$$P(\chi, \varepsilon) = -\frac{1}{2i\pi}\int\limits_{\Gamma} R(\zeta, \chi, \varepsilon).$$

Assumption 2.1.

$$dim P(\chi, \varepsilon) = dim P(0,0) = 1.$$

Proof of assumption 2.1 follows from lemma 1.4.10 [16].

Assumption 2.2. Let λ be a eigenvalue of operator $T(\chi, \varepsilon)$ corresponding to eigenvalue λ of operator T, then

$$\lambda(\chi, \varepsilon) = tr\left(T(\chi, \varepsilon)P(\chi, \varepsilon)\right) = \lambda + tr\left(T(\chi, \varepsilon) - \lambda\right)P(\chi, \varepsilon).$$

Proof of this assumption follows from definition and from

$$tr P(\chi, \varepsilon) = dim P(\chi, \varepsilon) = 1.$$

This formula gives a complete solution of a eigenvalues problem .

Assumption 2.3.

$$\lambda(\chi,\varepsilon)=-\frac{1}{2i\pi}tr\int\limits_{\Gamma}(\zeta-\lambda)R(\zeta,\chi,\varepsilon)d\zeta. \tag{3}$$

In fact,

$$\begin{aligned}&(T(\chi,\varepsilon)-\lambda)R(\zeta,\chi,\varepsilon)=(T(\chi,\varepsilon)-\zeta+\zeta)R(\zeta,\chi,\varepsilon)\\&=[(T(\chi,\varepsilon)-\zeta)+(\zeta-\lambda)]R(\zeta,\chi,\varepsilon)=J+(\zeta-\lambda)R(\zeta,\chi,\varepsilon),\end{aligned}$$

Integrating both parts of last parity on ζ, we obtain

$$-\frac{1}{2i\pi}\int\limits_{\Gamma}R(\zeta,\chi,\varepsilon)d\zeta=-\frac{1}{2i\pi}\int\limits_{\Gamma}d\zeta-\frac{1}{2i\pi}\int\limits_{\Gamma}(\zeta-\lambda)R(\zeta,\chi,\varepsilon)d\zeta,$$

as

$$\int\limits_{\Gamma}R(\zeta,\chi,\varepsilon)d\zeta=P(\chi,\varepsilon),\int\limits_{\Gamma}d\zeta=0$$

and Assumption 2.3 is proved.

Assumption 2.4. The parity

$$\lambda(\chi,\varepsilon)-\lambda=-\frac{1}{2i\pi}\int\limits_{\Gamma}\log\left\{[1+(\chi T'+\varepsilon T'')]\,R(\zeta,T)\right\}d\zeta$$

holds.

Proof. From formula (3) we have

$$\lambda(\chi,\varepsilon)-\lambda=-\frac{1}{2i\pi}tr\int\limits_{\Gamma}\sum_{p=0}^{\infty}\frac{1}{p}(\zeta-\lambda)\frac{d}{d\zeta}[-(\chi T^{'}+\varepsilon T^{''})R(\zeta,T)]^{p}d\zeta,$$

substitution instead of resolvent $R(\zeta,\chi,\varepsilon)$ of (2) give us

$$\begin{aligned}\lambda(\chi,\varepsilon)-\lambda\ &=-\tfrac{1}{2i\pi}tr\int\limits_{\Gamma}(\zeta-\lambda)R(\zeta,\chi,\varepsilon)d\zeta\\&=\tfrac{1}{2i\pi}\int\limits_{\Gamma}\log\left\{(1+[(\chi T'+\varepsilon T'')R(\zeta,T)])\right\}d\zeta.\end{aligned}$$

Till now we considered various power series of χ and ε and did not specify obviously a condition of their convergence. Now we investigate such conditions.

Series (2) obviously converges if the parity

$$\|\chi T'+\varepsilon T''\|<1$$

holds, and this condition takes place if

$$|\chi| < \frac{1}{2}||T'R(\zeta, T)||,\ |\varepsilon| < \frac{1}{2}||T''||R(\zeta, T).$$

Let's designate as $r_0(\chi)$ and $r_1(\varepsilon)$ values χ and ε for which we have

$$||T'R(\zeta, T)||^{-1} = 2|\chi|,\ ||T''R(\zeta, T)||^{-1} = 2\varepsilon.$$

Clearly that a series (2) converges in regular intervals on $\zeta \in \Gamma$, if

$$|\chi| \leq r_0 = \min_{\zeta\in\Gamma} r_0(\zeta),\ \varepsilon| < r_1 = \min_{\zeta\in\Gamma} r_1(\zeta).$$

Then, radius of convergence r_0 and r_1 depend on a contour Γ.

Assumption 2.5. Let $\rho = \max_{\zeta\in\Gamma} r_1|\zeta - \lambda|$, then

$$|\lambda(\chi, \varepsilon) - \lambda| < \frac{\beta}{2\pi}.$$

In fact, from (3) we have

$$\lambda(\chi, \varepsilon) - \lambda = tr\,[T(\chi, \varepsilon)P(\chi, \varepsilon)] = -\frac{1}{2\pi i} tr \int\limits_{\Gamma} (\zeta - \lambda)R(\zeta, \chi, \varepsilon)d\zeta,$$

then

$$\begin{array}{l} |\lambda(\chi, \varepsilon) - \lambda| = \frac{1}{2\pi} tr \int\limits_{\Gamma} (\zeta - \lambda)R(\zeta, \chi, \varepsilon)d\zeta \\ \leq \frac{\rho}{2\pi} tr \int\limits_{\Gamma} R(\zeta, \chi, \varepsilon)d\zeta = \frac{\rho}{2\pi} trP(\chi, \varepsilon). \end{array}$$

Since,

$$\frac{\rho}{2\pi} trP(\chi, \varepsilon) = dimP(\chi, \varepsilon) = 1,$$

then

$$|\lambda(\chi, \varepsilon) - \lambda| < \rho(2\pi)^{-1}. \tag{4}$$

The simple analysis of the formula (4) shows that the less ρ is the more closer $\lambda(\chi, \varepsilon)$ comes nearer to λ, and it should be. And still, than it is less $|\chi|, |\varepsilon|$ should be in general that speaking modules $|\chi|, |\varepsilon|$ that $\lambda(\chi, \varepsilon)$ also has got in Γ.

Assumption 2.6. Let $T'R(\zeta, T) \in G_F$:

$$||(\chi T' + \varepsilon T'')R(\zeta, T)|| < 1,$$

$$\{T'R(\zeta, T) + T''R(\zeta, T)\}\, v = \Sigma\,[a_k(v_i\varphi_i)\varphi_i + b_k(v_i\varphi_i)\psi_i]\,, \tag{5}$$

where

$$(\varphi_i\varphi_j) = 0,\ (\varphi_i\varphi_i) = 1,\ (\varphi_i\psi_j) = 0,$$

then

$$(T'R+T''R)^{2n}=(T'R)^{2n-1}+(T''R)^{2n-1},$$
$$(T'R+T''R)^{2n+1}=(T'R)^{2n+1}+T'(T'')^{2n}R^{2n+1}.$$

Proof. Let' consider a double power series

$$1+(-\chi T'R(\zeta,T))+(-\chi T'R(\zeta,T))+(-\chi T'R(\zeta,T))^2+(-\chi T'R(\zeta,T))^3+$$
$$+\cdots+(-\varepsilon T''R(\zeta,T))+(-\chi T'R(\zeta,T))(-\varepsilon T''R(\zeta,T))+$$
$$+(-\varepsilon T''R(\zeta,T))(-\chi T'R(\zeta,T))+\cdots+(-\chi T''R(\zeta,T))+(-\varepsilon T''R(\zeta,T))^2+\cdots$$

The general member of this series can be rewrote as

$$[(\chi T'+\varepsilon T'')(R(\zeta,T))]^p.$$

Let's rewrite formula (5),

$$\begin{aligned}
&[\chi T'R(\zeta,T)+\varepsilon T''R(\zeta,T)]v^2\\
&=\{[\chi T'R(\zeta,T)]^2+\chi\varepsilon T'R(\zeta,T)T''R(\zeta,T)+\chi\varepsilon T''R(\zeta,T)T'R(\zeta,T)\}v\\
&=\{\chi^2[T'R(\zeta,T)]^2\chi\varepsilon[T''R(\zeta,T)T'R(\zeta,T)]v;\\
&[\chi T'R(\zeta,T)+\varepsilon T''R(\zeta,T)]^{2n}v=[\chi^{2n}(T'R)^{2n}+\varepsilon\chi^{2n-1}T''R(T'R)^{2n-1}]v;\\
&(T'R(\zeta,T)+T''R(\zeta,T))^3v\\
&=(T'R(\zeta,T)+T''R(\zeta,T))^2*(T'R(\zeta,T)+T''R(\zeta,T))v\\
&=[(T'R)^3+T''R(T'R)^2]v;\\
&T'R+T''R]^{2n+1}v=[(T'R)^{2n}+T''(T')^{2n}(R(\zeta,T))^{2n+1}]v.
\end{aligned}$$

That's proves the Assumption 2.6.

By way of consequence it would be desirable to note one interesting fact.

Corollary 2.7. Let $T(x)=\chi T'+T$, $\lambda_n(x)$ are eigenvalues of $T(x)$, λ_n are eigenvalues of T, where $T'R(\zeta,T)$ and $R(\zeta,T)$ are quite continuous operators, χ is a complex parameter. Let

$$R(\zeta,T)T'=T'R(\zeta,T),$$

then

$$\lambda_n(\chi)-\lambda_n=\chi trT'P_n, \tag{6}$$

where [16]

$$P_n=-\frac{1}{2i\pi}\int\limits_{\Gamma_n}R(\zeta,T)d\zeta.$$

Proof. From

$$\lambda_n(\chi) - \lambda_n = \frac{1}{2i\pi} \int\limits_{\Gamma_n} tr \left(\sum \frac{(-1)^p}{p!} (\chi T' R)^p \right) d\zeta,$$

considering

$$\frac{d^{n+1}}{d\zeta^{n+1}} (R(\zeta, T)) = n! R(\zeta, T),$$

and

$$n![T' R(\zeta, T)]^n = [R(\zeta, T)]^n n! (T')^n,$$

we obtain

$$\lambda_n(x) - \lambda_n = \frac{1}{2i\pi} tr \sum \frac{(-\chi)^n}{nn!} (T')^n \frac{d^{n+1}}{d\zeta^{n+1}} R(\zeta, T) d\zeta.$$

Let's calculate integrals

$$\int\limits_{\Gamma_n} \frac{d}{d\zeta} (R(\zeta, T)) d\zeta = \int\limits_{\Gamma_n} d(R(\zeta, T)) = 0; n > 1,$$

therefore

$$\lambda_n(x) - \lambda_n = -\frac{1}{2i\pi} tr \int\limits_{\Gamma_n} \chi T' R(\zeta, T) d\zeta = \chi tr T' p_n.$$

Corollary 2.7 is proved. If T coincides with a projector, the statement of Corollary 2.7 is obvious without planimetric integration. As

$$\lambda_n(x) - \lambda_n = -\frac{1}{2i\pi} \int\limits_{\Gamma_n} tr[(\zeta - \lambda) R(\zeta, \chi, \varepsilon)] d\zeta$$

then

$$\lambda_n(x) - \lambda_n = \frac{1}{2i\pi} \int\limits_{\Gamma_n} tr(\sum [(\chi T' + \varepsilon T'') R(\zeta, T)]^P \frac{1}{P!}) d\zeta =$$

$$\frac{1}{2i\pi} \int\limits_{\Gamma_n} tr[\chi T' R(\zeta, T) + \varepsilon T'' R(\zeta, T) + \chi^2 (T')^2 (R(\zeta, T)))^2 +$$

$$\chi \varepsilon T' R(\zeta, T) T'' R(\zeta, T) + \chi \varepsilon T'' R(\zeta, T) T' R(\zeta, T) +$$

$$\varepsilon^2 (T'' R(\zeta, T))^2 + \cdots] d\zeta.$$

Theorem 2.8. Let operator T' be permutable with operator T, and operator $T''R(\zeta, T)$ is nilpotent operator with parameter of nilpotence equal to 2, $T'R, T''R \in G_F$

$$\lambda_n(x) - \lambda_n = \chi spT'p_n + \varepsilon spT''p_n$$

where p_n is a Riss's projector of operator T, corresponding to λ_n:

$$p_n = \frac{1}{2i\pi} \int\limits_{\Gamma_n} R(\zeta, T) d\zeta.$$

Proof. From (6),

$$\lambda_n(\chi, \varepsilon) - \lambda_n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \chi^m \varepsilon^n \widehat{\lambda}_{nm},$$

where

$$\widehat{\lambda}_{nm} = \frac{(-1)^{m+n+1}}{2i\pi} \int\limits_{\Gamma_n} a_{nm} tr(T'R(\zeta, T))^m (T''R(\zeta, T))^n d\zeta.$$

As operator $T''R(\zeta, T)$ nilpotent, then

$$\widehat{\lambda}_{0m} = \frac{(-1)^{m+1}}{2i\pi} C^k_{mk} \int\limits_{\Gamma_n} (T')^m (R(\zeta, T))^m,$$

$$\widehat{\lambda}_{1m} = \frac{(-1)^{m+2}}{2i\pi} C^m_{mk} \int\limits_{\Gamma_n} (T'')^m R(\zeta, T)(T')R^m,$$

$$\widehat{\lambda}_{jm} = \frac{(-1)^{m+1}}{2i\pi} \int\limits_{\Gamma_n} a_{jm} (T''R(\zeta, T))^m T'R.$$

We obtain

$$\widehat{\lambda}_{0m} = \chi trT'p_n.$$

Now as at proof of Corollary 2.7 we have

$$\lambda_n(\chi, \varepsilon) = \chi \widehat{\lambda}_{0m} + \varepsilon \widehat{\lambda}_{1m}.$$

For calculation $\widehat{\lambda}_{1m}$ we shall take advantage that

$$\int\limits_{\Gamma_n} d(R^2(\zeta, T)) = 0.$$

Thus

$$\widehat{\lambda}=\frac{(-1)^{m+2}}{2i\pi}a_{1m}\int\limits_{\Gamma_n}T'(R(\zeta,T))(T''R(\zeta,T))d\zeta=$$

$$\int\limits_{\Gamma_n}R(\zeta,T)T'T''R(\zeta,T)d\zeta=0.$$

Considering that

$$(\frac{d}{d\zeta})^n R(\zeta,T)=n![R(\zeta,T)]^{n+1},(n=1,2,3,\cdots)$$

We obtain

$$\int\limits_{\Gamma_n}T'T''R^{k+1}(\zeta,T)d\zeta=\begin{cases}T'p_n,\ k=0\\ 0,\ k\neq 0,\end{cases}$$

thus

$$\widehat{\lambda}_{0m}=\chi trT'p_n,\widehat{\lambda}_{1m}=\varepsilon trT''p_n$$

Theorem 2.8 is proved. Now using Theorem 2.8 we shall calculate eigenvalues of a problem

$$u''+cD^{1-\alpha}u+\lambda u=0,u(0)=0,u(\pi)=0,0<\alpha<1 \tag{7}$$

where $D^{1-\alpha}$ is the operator of fractional order $1-\alpha$ in Weil sense, i.e.

$$D^{1-\alpha}u=\frac{1}{2\pi}\frac{d}{dx}\int\limits_0^{2\pi}u(x-t)\psi^{1-\alpha}(t)dt,$$

$$\psi^{1-\alpha}(t)=2\sum_{k=1}^{\infty}\frac{\cos[kt-(1-\alpha)\pi/2]}{k^{1-\alpha}}.$$

Let's first $c(x)\equiv const$. Let's present the operator $D^{1-\alpha}R(\zeta,T)$ as sum

$$D^{1-\alpha}R(\zeta,T)=\chi T'R(\zeta,T)+\varepsilon T''R(\zeta,T)$$

so that T' and T'' satisfied to conditions of Theorem 2.8. Since operator

$$Tu=\begin{cases}-u'',\\ u(0)=0,\ u(\pi)=0,\end{cases}$$

is full self-adjoined operator, then

$$R(\zeta,T)v=\sum_{k=1}^{\infty}\frac{2}{\pi}\frac{(v,\sin kx)}{k^2-\zeta}\sin kx.$$

Thus

$$D^{1-\alpha}R(\zeta,T)v = \frac{2}{\pi}D^{1-\alpha}\left[\sum_{k=1}^{\infty}\frac{2}{\pi}\frac{(v,\sin kx)}{k^2-\zeta}\sin kx\right]$$

As

$$\begin{aligned} D^{1-\alpha}\sin kx &= \frac{d}{dx}[D^{1-\alpha}\sin kx] = \frac{d}{dx}\left[n^{-\alpha}\sin(nx-\tfrac{\alpha\pi}{2})\right] \\ &= n^{1-\alpha}\left[\cos nx\cos\tfrac{\alpha\pi}{2}+\sin\tfrac{\alpha\pi}{2}\sin nx\right], \end{aligned}$$

then we obtain

$$\begin{aligned} D^{1-\alpha}R(\zeta,T)v &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{n^{1-\alpha}(v,\ \sin kx)}{k}\left[\cos kx\cos\tfrac{\alpha\pi}{2}+\sin\tfrac{\alpha\pi}{2}\right] \\ &= \frac{2}{\pi}\cos\tfrac{\alpha\pi}{2}\sum_{k=1}^{\infty}\tfrac{(v,\ \sin kx)}{k^2-\zeta}\cos kx+\frac{2}{\pi}\sin\tfrac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\tfrac{(v,\ \sin kx)}{k^2-\zeta}. \end{aligned}$$

Thus

$$D^{1-\alpha}R(\zeta,T)v = A\widetilde{R}(\zeta,T)v + B\widetilde{R}(\zeta,T)v$$

where

$$Av = \sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}(v,\sin kx)\sin kx,$$

$$Bv = \cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}(v,\sin kx)\cos kx.$$

Really

$$\begin{aligned} AR(\zeta,T)v &= \frac{2}{\pi}\sin\tfrac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\left[\sum_{k=1}^{\infty}\frac{(v,\ \sin kx)}{k^2-\zeta}\sin jx\sin kx\right]\sin kx \\ &= \frac{2}{\pi}\sin\tfrac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\frac{(v,\ \sin kx)}{k^2-\zeta}; \end{aligned}$$

$$BR(\zeta,T)v = \frac{2}{\pi}\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\left(\sum_{k=1}^{\infty}\frac{(v,\sin jx)}{k^2-\zeta}\sin jx,\sin jx\right)\cos kx = \frac{2}{\pi}\cos\frac{\alpha\pi}{2},$$

$$[AR+BR]^{2n+1}v = \left[(AR)^{2n+1}+BA^{2n}R^{2n+1}\right]v.$$

That that operators $A, R(\zeta,T)$ permutable is checked directly

$$\begin{aligned} AR(\zeta,T)v &= \frac{2}{\pi}\cos\tfrac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\left(\sum_{j=1}^{\infty}\frac{(v,\ \sin jx)}{j^2-\zeta}\sin jx,\sin kx\right)\sin kx \\ &= \frac{2}{\pi}\sin\tfrac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\frac{(v,\ \sin kx)}{k^2-\zeta}\sin kx, \end{aligned}$$

$$\begin{aligned} R(\zeta,T)Av &= \frac{2}{\pi}\cos\tfrac{\alpha\pi}{2}\sum_{n=1}^{\infty}\frac{(Av,\ \sin nx)}{n^2-\zeta}\sin nx \\ &= \frac{2}{\pi}\sin\sum_{n=1}^{\infty}\frac{\sum\limits_{k=1}^{\infty} n^{1-\alpha}(v,\ \sin kx)\sin nx}{n^2-\zeta}\sin nx \\ &= \frac{2}{\pi}\tfrac{\sin\alpha\pi}{2}\sum_{n=1}^{\infty}\frac{n^{1-\alpha}(v,\ \sin kx)}{n^2-\zeta}\sin nx = AR(\zeta,T)v, \end{aligned}$$

i.e.,

$$AR(\zeta,T)v = R(\zeta,T)Av.$$

Now, let's show nilpotence of operator $BR(\zeta,T)$

$$\begin{aligned} \left[BR(\zeta,T)\right]^2 v &= BR(\zeta,T)BR(\zeta,T)v \\ &= \frac{2}{\pi}\cos\tfrac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{1-\alpha}\frac{(BR(\zeta,T)v,\ \sin kx)}{k^2-\zeta}\cos kx \\ &= \frac{2}{\pi}\cos\tfrac{\alpha\pi}{2}\sum_{k=1}^{\infty}\frac{\left(\frac{2}{\pi}\cos\frac{\alpha\pi}{2}\sum\limits_{k=1}^{\infty}\frac{(v,\ \sin jx)}{j^2-\zeta}\cos jx,\ \sin kx\right)}{k^2-\zeta}\cos kx \\ &= \left[\frac{2}{\pi}\cos\tfrac{\alpha\pi}{2}\right]^2\sum_{k=1}^{\infty}k^{1-\alpha}\frac{\left(\sum\limits_{k=1}^{\infty}\frac{(v,\ \sin jx)}{j^2-\zeta}\cos jx,\ \sin kx\right)\cos kx}{k^2-\zeta} = 0. \end{aligned}$$

According to Theorem 1.4.1 [16], for eigennumbers of a problem $\mu_k = n^2 + \widehat{\lambda}_{0m} + \widehat{\lambda}_{1m}$. Let's calculate

$$\widehat{\lambda}_{0m} = trAp_n,$$

as

$$P_n = \frac{1}{2i\pi}\int\limits_{\Gamma_n} R(\zeta,T)d\zeta,$$

then p_n is integrated operator with a kernel

$$P_n(x,y) = \frac{2}{\pi}\sin nx\sin ny,$$

i.e.,

$$P_n v = \frac{2}{\pi}\int\limits_0^{\pi}\sin nx\sin ny\cdot v(y)dy = \frac{2}{\pi}(v,\sin nx)\sin nx,$$

we obtain

$$\begin{aligned} Ap_n v &= \tfrac{2}{\pi}\sin\tfrac{\alpha\pi}{2}\sum k^{1-\alpha}(v,\sin nx)(\sin nx,\sin kx)\sin kx \\ &= \tfrac{2}{\pi}\sin\tfrac{\alpha\pi}{2}(v,\sin nx)n^{1-\alpha}\sin nx. \end{aligned}$$

Let's find eigennumbers of operator AP_n

$$^{A}P_n v = \lambda v$$

or

$$\frac{2}{\pi}\sin\frac{\alpha\pi}{2}n^{1-\alpha}(v,\sin nx) = v\lambda.$$

Thus

$$\lambda_1 = n^{1-\alpha}\frac{2}{\pi}\sin\frac{\alpha\pi}{2},$$

then

$$AP_n = \lambda_1 = \frac{2}{\pi}n^{1-\alpha}\sin\frac{\alpha\pi}{2}.$$

Further

$$\begin{aligned} BP_n \quad &= \tfrac{2}{\pi}\sin\tfrac{\alpha\pi}{2}\textstyle\sum(p_n v,\sin nx)\sin nx \\ &= \tfrac{2}{\pi}\sin\tfrac{\alpha\pi}{2}\textstyle\sum\limits_{k=1}^{\infty} n^{1-\alpha}(\tfrac{2}{\pi}(v,\sin kx)\sin kx,\sin nx)\cos nx \\ &= \tfrac{2}{\pi}\sin\tfrac{\alpha\pi}{2}(v,\sin kx)\cos nx \\ &= \tfrac{2}{\pi}\sin\tfrac{\alpha\pi}{2}\cos nx\textstyle\int\limits_0^{\pi} v(t)\sin ntdt, \end{aligned}$$

i.e.,

$$BP_n v = \frac{2}{\pi}\sin\frac{\alpha\pi}{2}\cos nx\int\limits_0^{\pi} v(t)\sin ntdt.$$

Let's calculate a trace of operator BP_n

$$BP_n = \sum\lambda_n(BP_n).$$

To find eigennumbers of operator BP_n we shall solve the equation

$$BP_n v = v\lambda$$

or

$$\frac{2}{\pi}\sin\frac{\alpha\pi}{2}\cos nx\int\limits_0^{\pi} v(t)\sin ntdt = \lambda v(t).$$

Clearly that

$$BP_n = 0.$$

Thus

$$\mu_n = n^2 + n^{1-\alpha}\sin\frac{\pi\alpha}{2}.$$

For solution of a problem (7) in case $c(x) = const$ we use following theorem having certainly independent interest.

3. Estimations for eigenvalues

Theorem 3.1. Let eigennumbers of the self-adjoined operator A_0 with a discrete spectrum $\lambda_n^{(0)} = n^q, (n = 1, 2, 3, .., q)$, are normalized eigenvectors φ_n and let B be a closed limited operator, and

$$D_{A_0} \subset D_B; ||B\varphi_n|| = O(n^\rho)$$

then for eigennumbers λ_n of operator $A = A_0 + B$, we have

$$\lambda_n - n^q = O(n^\rho).$$

Proof. Let Γ_n be circle with the radius 1 and the center in point n^q, and the operator A is set by the formula $A = A_0 + B$. If n is a big enough number that inside of Γ_n for which other eigennumbers does not contain except for λ_n. The following known result is required to us.

Lemma 3.1. Let A_0 be a operator with compact resolvent $R(\zeta, A_0)$, and the operator B is compact rather A_0. If Δ is the limited area with straightened border Γ and

$$\sigma(A_0 + tB) \cap \Gamma = \emptyset$$

for all $t \in [0, 1]$, then the operator A also has a compact resolvent and

$$N(\Delta, A_0 + B) = N(\Delta, A_0),$$

where $N(A)$ is a number of eigenvalues of operator, laying inside the area.

Proof. Let $R(\zeta, A_0)$ be a resolvent of the operator A_0, then

$$R(A_0, \zeta) = \sum_{n=1}^{\infty} \frac{(v, \varphi_n)\varphi_n}{n^q - \zeta}.$$

By Lemma 3.1 follows, if $\|BR(\zeta, A_0)\|_{\Gamma_n} < 1$, then for all $t \in [0, 1]$, $\sigma(A_0 + tB) \cap \Gamma = \emptyset$, we have

$$\begin{aligned} ||BR(\zeta, A_0)||_{\Gamma_n} &= \| \sum_{n=1}^{\infty} \frac{(v, \varphi_n)\varphi_n}{k^q - \zeta} B\varphi_k \|_{\Gamma_n} \leq \sum_{n=1}^{\infty} \| \frac{(v, \varphi_n)\varphi_n}{k^q - \zeta} \|_{\Gamma_n} \|B\varphi_k\| \\ &\leq \frac{||(v, \varphi_n)\varphi_n||}{|k^q - \zeta|} n^\rho \leq \sum_{n=1}^{\infty} \frac{k^\rho}{k^\rho - (n^q - 1/3)}. \end{aligned}$$

Further

$$\sum_{n=1}^{\infty} \frac{k^\rho}{k^\rho - (n^q - 1/3)} = \sum_{n=1}^{n-1} \frac{k^\rho}{k^\rho - (n^q - 1/3)} + \frac{n^\rho}{n^\rho - n^q + 1/3} + \sum_{k=n+1}^{n-1} \frac{k^\rho}{k^\rho - (n^q - 1/3)}.$$

Last sum will be less than 1 if finite disturbance of the operator A_0 replaces on the operator $\widetilde{A_0}v = A_0 v - 3n^{\rho}(v, \varphi_n)\varphi_n$.

Theorem 3.2. Much more strengthens and generalizes known result of M.K. Gavurin.

Theorem of Gavurin. Let eigennumbers of the self-adjoined operator with a discrete spectrum $\lambda_n^{(0)} = n^q (n = 1, 2, 3, \cdots ; q > 1)$, normalized eigenfunctions φ_n and B are closed symmetric operator, and

$$D_{A_0} \subset D_B ; ||B\varphi_n|| = O(n^{\alpha}),$$

then for the eigennumbers operator

$$Av = A_0 v + Bv,$$

we have

$$\lambda_n = n^q \quad + \sum_{k \neq n} \frac{(B\varphi_k, \varphi_n)|(B\varphi_k, \varphi_n)|^2}{|k^q - n^q|^2} - \frac{\sum\limits_{k \neq n} \frac{|(B\varphi_k, \varphi_n)|^2}{|k^q - n^q|^2}}{1 + \sum\limits_{k \neq n} \frac{|(B\varphi_k, \varphi_n)|^2}{|k^q - n^q|^2}}$$

$$+ \frac{\sum\limits_{k, l \neq n} \frac{(B\varphi_k, \varphi_n)\overline{(B\varphi_n, \varphi_l)}}{(k^q - n^q)(l^q - n^q)}}{1 + \sum\limits_{k \neq n} \frac{|(B\varphi_k, \varphi_n)|^2}{|k^q - n^q|^2}} + O(n^{q-5}).$$

Here we shall note that, from Theorem 3.1, the known formula of I.M. Lifshits have greater appendices in quantum statistics follows.

Theorem of I.M. Lifshits. Let H be a self-adjoined hypermaximal operator, T is a self-adjoined finite operator, then

$$\sum (\mu_n - \lambda_n) = spT,$$

where μ_n is a eigenvalues of operator $H + T$, $\lambda_n = \lambda_n(H)$. From theorem 3.1, the following theorem generalizes Lifshits's theorem.

Theorem 3.3. In conditions of theorem 3.1 the following formula

$$\sum_{n=1}^{\infty} (\lambda_n(\chi, \varepsilon) - \lambda_n) = \varepsilon spT' + \chi T''$$

holds. Notice that the self-adjoining of operators T' and T'' are not supposed and under after it is possible to mean any regularization usual concept of a trace. Now in a case of $c(x) \neq const$ from theorem 3.2 follows next

Theorem 3.4. For eigennumbers λ_n of a problem estimations are fair

$$\lambda_n - n^2 = O(n^\alpha).$$

Proof. Proof of this theorem follows from proof of Theorem 3.2 for $q = 2$.

4. **Caushy Problem for differential equations with fractional derivatives.**

Let's consider a problem

$$-u'' + D^\alpha_{0x} u + \lambda u = 0 \tag{1}$$

$$u(0) = 0, u'(0) = 1. \tag{2}$$

In [63], the problem (1)-(2) has an unique solution $u(x, \lambda)$ for any λ. For the further it will be important to know whether function $u(x, \lambda)$ will be whole function of a zero kind.

Theorem 4.1. the Solution $u(x, \lambda)$ of the problem (1)-(2) is the whole function of a zero kind of parameter λ.

Proof. Assume that $u_0 = x$, let for

$$u_n(x, \lambda) = u_0(x, \lambda) + \int\limits_0^x \{(x-t)^{1-\alpha} u_{n-1}(t, \lambda) + \lambda(x-t) u_{n-1}(t, \lambda)\} dt.$$

Let $|\lambda| \leq N$, then

$$| \, u_1(x, \lambda) - u_0(x, \lambda) \, | \leq \frac{x^{2-\alpha}}{2-\alpha} + N\frac{x^2}{2}.$$

For $n \geq 2$, we have

$$| \, u_2(x, \lambda) - u_0(x, \lambda) \, | \leq | \, \{ \int\limits_0^x \{(x-t)^{1-\alpha} u(t) + \lambda(x-t) u_1(t) - (x-t)^{1-\alpha} t - \lambda(x-t) t\} dt$$

From here

$$\begin{aligned} | \, u_2(x, \lambda) - u_1(x, \lambda) \, | &\leq \frac{x^{3-\alpha}}{(2-\alpha)(3-\alpha)} + N\frac{x^3}{3!} + N\frac{x^{3-\alpha}}{(2-\alpha)(3-\alpha)} \\ &= (N+1)\frac{x^{3-\alpha}}{(2-\alpha)(3-\alpha)} \end{aligned}$$

and in general

$$| \, u_n(x, \lambda) - u_{n-1}(x, \lambda) \, | \leq (N+M)^n \left[\frac{x^{n-\alpha}}{(2-\alpha)(3-\alpha)\cdots(n-\alpha)} + N\frac{x^n}{n!} \right].$$

Hence for $|\lambda| \leq N$ and $0 \leq x \leq 1$, a series

$$u(x,\lambda) = u_0(x,\lambda) + \sum_{n=1}^{\infty}\{u_n(x,\lambda) - u_{n-1}(x,\lambda)\} \tag{3}$$

converges in regular intervals on λ. As for any $n \geq 2$

$$\begin{aligned} u_n'(x,\lambda) - u_{n-1}'(x,\lambda) &= \int_0^x \{(x-t)^{-\alpha}(u_{n-1}(t,\lambda) - u_{n-2}(t,\lambda)) \\ &+ \lambda(u_{n-1}(t,\lambda) - u_{n-2}(t,\lambda))\}dt, \\ u_n''(x,\lambda) - u_{n-1}''(x,\lambda) &= \{D_{0x}^{\alpha} + \lambda\}(u_{n-1}(x,\lambda) - u_{n-2}(x,\lambda)), \end{aligned}$$

then series, obtained by unitary and two-multiple differentiation of series (3) also converge in regular intervals on x. Thus

$$\begin{aligned} u''(x,\lambda) &= \sum_{n=1}^{\infty}\{u_n''(x,\lambda) - u_{n-1}''(x,\lambda)\} \\ &= u_1''(x,\lambda) - u_0''(x,\lambda) + \sum_{n=2}^{\infty}\{u_n''(x,\lambda) - u_{n-1}''(x,\lambda)\} \\ &= \{D_{0x}^{\alpha} + \lambda\}\{u_0(x,\lambda)\} + \sum_{n=2}^{\infty}\{u_n(x,\lambda) - u_{n-1}(x,\lambda)\} \\ &= \{D_{0x}^{\alpha} + \lambda\}\{u_0(x,\lambda)\} + \lambda u_0(x,\lambda), \end{aligned}$$

and $u(x,\lambda)$ satisfied to equation (1). Clearly that $u(x,\lambda)$ to conditions (2).

Now, with a problem (1)-(2), we will consider next problem

$$v'' - v'(x) = \lambda v(x)$$

$$v(0) = 0, v'(0) = 1.$$

It is known that the solution $v(x,\lambda)$ of this problem is the whole function of a zero kind. Let $v_0(x,\lambda)$, and for

$$v_n(x,\lambda) = v_0(x,\lambda) + \int_0^x \{1 + \lambda(x-t)v_{n-1}(t,\lambda)\}dt.$$

Let's rewrite $u_n(x,\lambda)$ and $v_n(x,\lambda)$ like

$$u_n(x,\lambda) = a_0(x) + \lambda a_1(x) + \cdots + \lambda^n a_n(x)$$

$$v_n(x,\lambda) = b_0(x) + \lambda b_1(x) + \cdots + \lambda^n b_n(x)$$

Clearly that

$$|a_i(x)| < b_i(x), \quad x \in (0,1).$$

Then, due to theorem of Adamar, $u(x,\lambda)$ is a whole function of a zero kind.

5. **About one method estimating of first eigenvalues.**

Let' consider a problem

$$-u'' + D_{0x}^{\alpha} u + \lambda u = 0 \tag{1}$$

$$u(0) = 0, u'(0) = 1 \tag{2}$$

Let $u(x,\lambda)$ be a solution of a problem (1)-(2). We have already established that $u(x,\lambda)$ is a whole function of a zero kind. Hence it is possible to present like infinite multiplication

$$u(x,\lambda) = c \prod_{j=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_j}\right),$$

where c is a while unknown constant, λ_j are zeros of function $u(1,\lambda)$. Since zeros of function $u(1,\lambda)$ coincide with own values of a problem (1)-(2)

$$-u'' + D_{0x}^{\alpha} u = \lambda u, \tag{1'}$$

$$u(0) = 0, u(1) = 0, \tag{2'}$$

then studying of eigenvalues of a problem (1′)-(2′) is reduced to studying of zeros of function $u(1,\lambda)$. First of all one interesting statement is required to us.

Lemma 5.1. All eigenvalues of a problem (1′)-(2′) are positive.

Proof. For proof of the given statement we shall consider a following problem

$$-u'' + D_{0x}^{\alpha} u = \lambda u$$

$$u(0) = 0, u(0) = 0.$$

This problem is equal to equation

$$u(x) = x + \int_0^x \left[(x-t)^{1-\alpha} - \lambda(x-t)\right] u(t)dt. \tag{3}$$

Let's designate

$$K_1(x,t) = \begin{cases} (x-t)^{1-\alpha}, \ 0 \le t < x < 1, \\ 0, \ x < t \le 1, \end{cases}$$

also we shall define further sequence of kernels $\{K_n(x,t)\}_1^\infty$ by means of recurrent parities

$$K_{n+1}(x,t) = \int\limits_t^x K_n(x,t_1)K_1(x,t_1)dt_1.$$

Elementary calculations show that

$$K_2(x,t) = \lambda\frac{\Gamma(1-\alpha)\Gamma(2-\alpha)}{\Gamma(4-2\alpha)}(x-t)^{3-2\alpha} - 2\lambda\frac{\Gamma(2)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}(x-t)^{3-\alpha} + \frac{1}{3!}(x-t)^3,$$

using an induction on n we obtain

$$K_{n+1}(x,t) = K^0_{n+1}(x,t) - \lambda K^1_{n+1}(x,t) + \lambda K^2_{n+1}(x,t) + \cdots, \quad x \geq t.$$

Elementary calculations show that

$$K^i_{n+1}(x,t) \geq 0$$

for $x \geq t$ for any n. From here for resolvent of equation (3) we have a formula

$$R(x,\lambda,t) = \sum_{n=0}^{\infty} K_{n+1}(x,t,\lambda) = a_0(x,t) - \lambda a_0(x,t) + \lambda^2 a_2(x,t) + \cdots$$

where $a_i(x,t) \geq 0$ for $x \geq t$. As a solution (3) is represented as

$$u(x) = x + \int\limits_0^x R(x,\lambda,t)tdt$$

then

$$u(1,\lambda)1 + \int\limits_0^1 R(1,\lambda,t)tdt = \widetilde{a_0} - \lambda\widetilde{a_1} + \lambda^2\widetilde{a_2} - \lambda^3\widetilde{a_3} + \cdots$$

where $\widetilde{a_0}, \widetilde{a_1}, \cdots, \widetilde{a_n}, \cdots$ are nonnegative numbers. Thus λ is a eigenvalue of a problem (1)-(2), iff λ is a zero of function

$$w(\lambda) = u(1,\lambda) = \widetilde{a_0} - \lambda\widetilde{a_1} + \lambda^2\widetilde{a_2} - \cdots$$

But it is obvious that $w(\lambda)$ hasn't negative zeros.

From estimations obtained in (1) follows that eigenvalues $\{\lambda_j\}$ of a problem satisfy to a parity $\sum\limits_{j=1}^{\infty} \lambda_j^{-1} < \infty$. Therefore for operator A induced by the differential equation (1') and regional conditions (2') we can enter a determinant

$$D_a(\lambda) = \prod_{j=1}^{\infty}(1 - \lambda\lambda_j^{-1}(A))$$

Clearly that

$$D_a(\lambda) = u(1, \lambda).$$

Further as well as in [3] we shall consider a logarithmic derivative of function

$$[\ln(D_A(\lambda))]' = \frac{D'_A(\lambda)}{D_A(\lambda)} = -\sum_{n=1}^{\infty} \chi_{n+1}\lambda^n. \tag{4}$$

where $\chi_n = spurA^n$. By virtue of a parity (4) we have

$$\frac{u'(\lambda, 1)}{u(\lambda, 1)} = -\sum \chi_{n+1}\lambda^n.$$

Let Taylor's series $u(x, \lambda)$ look like

$$u(x, \lambda) = \sum_{n=0}^{\infty} S_n(x)\lambda^n$$

where $u(x, \lambda)$ is a solution of a problem (1')-(2') then

$$\sum \frac{nS_n(1)\lambda^{n-1}}{S_n\lambda^n} = \sum \chi_{n+1}\lambda^n.$$

From last parity we have

$$\chi_{n+1} + \sum_{m=1}^{n} S_m(1)\chi_{n+1-m} = -(n+1)\lambda^n$$

These equality form system of the non-uniform linear equations for χ_{n+1}

$$S_0(1)\chi_1 + 0 * \chi_2 + 0 * \chi_3 + \cdots + 0 * \chi_n = S_1$$

$$S_1(1)\chi_1 + S_0\chi_2 + 0 * \chi_3 + \cdots + 0 * \chi_n = S_2$$

$$\cdots\cdots\cdots\cdots$$

$$S_{n-1}(1)\chi_1 + S_{n-2}\chi_2 + \cdots + S_0(1)\chi_n = -nS_n$$

From here for definition χ_j we have a recurrent formulas

$$\chi_n = -\left[nS_n(1) + \sum_{p=1}^{n-1} S_{p-1}(1)\chi_{n-p}\right] \tag{5}$$

From (5) follows that for definition of $\chi_n (n = 1, 2, 3, ..)$ it is enough to know that $S_n(1)(n = 1, 2, 3, ..)$. We know that the problem

$$-u'' + D_{0x}^{\alpha} u = \lambda u, u(0) = 0, u(1) = 1$$

equals to the equation

$$u(x) = x + \int\limits_0^x \{(x-t)^{1-\alpha} - \lambda(x-t)\} dt \tag{6}$$

As a solution (6) is the whole function of parameter λ then

$$u(x, \lambda) = \sum_{n=0}^{\infty} S_n(x) \lambda_n, \tag{7}$$

substituting (7) in (6) we have

$$S_0(x)+\lambda S_1(x)+\lambda^2 S_2(x)+\cdots = x+\int\limits_0^x \left[(x-t)^{1-\alpha} - \lambda(x-t)\right] \left[S_0(t) + \cdots + \lambda^n S_n(t) + \cdots\right] dt. \tag{8}$$

From (8) follows

$$S_0(x) = x + \int\limits_0^x (x-t)^{1-\alpha} u(t) dt, \tag{9}$$

$$S_1(x) = -\int\limits_0^x (x-t) S_0(t) dt + \int\limits_0^x (x-t)^{1-\alpha} S_1(x) dt, \tag{10}$$

$$S_2(x) = \int\limits_0^x (x-t)^{1-\alpha} S_2(x) dt + \int\limits_0^x (x-t)^{1-\alpha} S_1(x) dt, \tag{11}$$

$$\cdots\cdots\cdots\cdots$$

Solving equation (9) we obtain

$$S_0(x) = x E_\rho(x^{1/\rho}; 2); \quad S_0(1) = 1.$$

For solution of equation (10) we shall calculate

$$\int\limits_0^x (x-t) S_0(t) dt = \int\limits_0^x t E_\rho(t^{1/\rho}; 2)(x-t) dt = x^3 E_\rho(x^{1/\rho}; 4)$$

then

$$\begin{aligned} a_0(x) &= x^3E_\rho(x^{1/\rho};4)+\int\limits_0^x(x-t)^{1/\rho-1}E_\rho((x-t)^{1/\rho};\frac{1}{\rho})t^3E_\rho(t^{1/\rho};4)dt \\ &= -c_1(x^3E_\rho(x^{1/\rho};4)+c_0\rho x^{3+1/\rho}[E_\rho(x^{1/\rho};3+1/\rho)-(3+1/\rho)E_\rho(x^{1/\rho};4+1/\rho)]). \end{aligned}$$

It is similarly possible to show, that

$$\begin{aligned} a_2(x)= &- cx^5E_\rho(x^{1/\rho};6)+c\rho x^{5+1/\rho}E_\rho(x^{1/\rho};5+1/\rho) \\ &- c\rho(5+1/\rho)x^{5+1/\rho}E_\rho(x^{1/\rho};6+1/\rho) \\ &- c\Big[\rho x^{5+1/\rho}E_\rho(x^{1/\rho};5+1/\rho)-\rho(5+1/\rho)x^{5+1/\rho}E_\rho(x^{1/\rho};6+1/\rho)\Big] \\ &+ c\rho\Big\{\rho x^{2/\rho+5}E_\rho(x^{1/\rho};5+1/\rho)x^{2/\rho+5}E_\rho(x^{1/\rho};6+1/\rho) \\ &- c\rho(5+1/\rho)\Big[x^{2/\rho+5}E_\rho(x^{1/\rho};6+1/\rho)-(5+2/\rho)E_\rho(x^{1/\rho};6+1/\rho)x^{2/\rho+5} \\ &+ \frac{1}{2\rho}\sum_{k=0}^{\infty}x^{(k+2)/\rho-5}\frac{k^2}{\Gamma(\frac{k}{\rho}+6+\frac{2}{\rho})}\Big]\Big\}. \end{aligned}$$

Now, let's note that from (5) follows that

$$S_1(1)=-\chi_1, S_2(1)=\frac{1}{2}(\chi_1^2-\chi_2).$$

As all eigenvalues of a problem $(1')-(2')$ are positive then obviously

$$\lambda_1>\frac{1}{\chi_1}=-\frac{1}{S_1(1)}.$$

The estimation from below for λ_1 looks like

$$\lambda_1<\frac{\chi_1}{\chi_2}$$

Now if to consider that S_1 and S_2 it is possible to within to count up 10^{-2} that we shall obtain

Theorem 5.1. For first eigennumber λ_1 of a problem $(1')-(2')$ we have a parity

$$(1.85)^{-1}<\lambda_1<3.86.$$

Let's note that it is similarly possible to find estimations for eigenvalues of a problem

$$-u''+D_{0x}^{\alpha_0}u+D_{0x}^{\alpha_1}u=\lambda u,$$
$$u(0)=0,\ u(1)=0.$$

6. Mutually adjoined problems and questions of completeness of eigenfunctions.

For equation

$$u'' + \sum_{i=1}^{n} a_i(x) D_{0x}^{\alpha_i} \omega_j(x) u = \lambda u, \ 0 < \alpha_i < 1, \tag{1}$$

let's consider a problem

$$u(0) = 0, \ u(1) = 0. \tag{2}$$

With a problem (1)-(2), we consider a problem

$$z'' + \sum_{i=1}^{n} \omega_j(x) (D_{0x}^{\alpha_i})^* a_i z + \lambda z = 0, \tag{3}$$

$$z(0) = 0, \ z(1) = 0,$$

here $(D_{0x}^{\alpha_i})^*$ is adjoined operator to operator $D_{0x}^{\alpha_i}$ i.e.

$$D_{0x}^{\alpha_i})^* u = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^1 \frac{u(t)}{(t-x)^{\alpha}} dt,$$

in the certain sense associate with a problem (1)-(2). These problems we shall call as mutually adjoined

Let

$$\{v_n(x)\}_{n=1}^{\infty}$$

is a system of eigenfunctions of a problem (1)-(2), and $\{z_n(x)\}_{n=1}^{\infty}$ is a system of eigenfunctions of a problem (3)-(4).

We shall establish that a system of functions

$$\{v_n(x)\}_{n=1}^{\infty}, \quad \{z_n(x)\}_{n=1}^{\infty}$$

is biorthogonal on $[0, 1]$.

Theorem 6.1. System of eigenfunctions

$$\{v_n(x)\}_{n=1}^{\infty}$$

is full in $L_2(0, 1)$.

Proof. Let's enter operator

$$Mu = \begin{cases} -u'', \\ u(0) = 0, \ u(1) = 0, \end{cases}$$

then a problem (1)-(2) is equal to equation

$$u + M^{-1}\left\{\sum_{i=1}^{n} a_i(x) D_{0x}^{\alpha_i}\omega_i(x)u\right\} - \lambda M^{-1}u = 0,$$

i.e., a problem has communicated to research of operators of Keldysh's type, where

$$M^{-1}u = \int_0^1 G_T(x,t)u dt,$$

$$G_T(x,t) = \begin{cases} t(x-1), & t \leq x, \\ x(t-1), & t \geq 1. \end{cases}$$

Clearly that M^{-1} is a complete self-adjoined operator,

$$M^{-1}\left(\sum_{i=1}^{n} a_i(x) D_{0x}^{\alpha_i}\omega_i(x)u\right) \in G_1.$$

M^{-1} is a kernel operator. Then, from theorem of Keldysh [3], follows corresponding completeness. Let's designate $n(r)$ an exact number of eigenvalues of a problem (1)-(2) laying in circle $|\lambda| < r$.

Problems about distribution of eigenvalues it is put as research of asymptotical properties for $n \to \infty$ of value $n(r)$.

Theorem 6.2. The parity is fair

$$\lim_{r \longrightarrow \infty} n(r)/r^{1/2} = 1.$$

Proof. Studying of a spectrum of a problem is reduced to studying a spectrum of a linear operational bunch

$$L(\lambda) = J + M^{-1}\left\{\sum_{i=1}^{n} a_i(x) D_{0x}^{\alpha_i}\omega_i\right\} - \lambda M^{-1}.$$

Clearly

$$M^{-1}\left(\sum_{i=1}^{n} a_i(x) D_{0x}^{\alpha_i}\omega_i(x)u\right) \in G_1.$$

M^{-1} is a positive operator. If for function of distribution $rn(r)$ is possible to pick up not decreasing function $\varphi(r), (0 \leq r \leq \infty)$, possessing properties

1. $\lim_{r \longrightarrow \infty} \varphi(r) = \infty, \quad \varphi(r) \uparrow (r \longrightarrow \infty)$;
2. $\lim_{r \longrightarrow \infty} [\ln(\varphi(r))]' = \lim_{r \longrightarrow \infty} \frac{\varphi'(r)}{\varphi(r)} < \infty$;
3. $\lim_{r \longrightarrow \infty} n(r)/\varphi(r) = 1$.

Then by theorem of Keldysh,

$$\lim_{r \longrightarrow \infty} n(r)/n(r, M^{-1}) = 1.$$

As $\varphi(r)$ in our case, obviously, it is possible to take function

$$\varphi(r) = r^{1/2}.$$

This proves Theorem 6.2.

Let's consider a differential expression

$$Ly = -y'' + D^{\alpha}_{0x} y$$

on finite interval $[0, 1]$.

Let T be operator, certain in Hilbert space $H = L_2(0, 1)$ on operator L with regional conditions $y(0) = y(1) = 0$. Certainly, operator T is weak disturbance of operator

$$Au = \begin{cases} u'' \\ u(0) = 0, \ u(1) = 0 \end{cases}$$

Let's formulate the theorem rather important at studying operators of a kind T.

Theorem 6.3. Operator T is dissipative.

Proof.

$$(Ty, y) = \left(-\frac{d^2}{dx^2} y, y\right) + (D^{\alpha}_{0x} y, y);$$

$$(D^{\alpha}_{0x} y, y) \geq 0,$$

then operator T is dissipative.

7. Construction of one biorthogonal system

Let's consider the Storm-Liouville problem of fractional differential equation

$$\begin{cases} u'' + a_m(x) u' + a_i(x) D^{\alpha_i}_{0x} \omega_i(x) u + \lambda u = 0, \\ u(0)\cos(\alpha) + u'(0)\sin(\alpha) = 0, \\ u(1)\cos(\beta) + u'(1)\sin(\beta) = 0 \end{cases} \tag{1}$$

and the problem

$$\begin{cases} z - [a_m(x) z(x)]' - \omega_i(x) D^{\alpha_i}_{x1} a_i(t) z(t) + \lambda z = 0, \\ z(0)\cos(\alpha) + z'(0)\sin(\alpha) = 0, \\ z(1)\cos(\beta) + z'(1)\sin(\beta) = 0. \end{cases} \tag{2}$$

We shall call problems (1) and (2) are mutually adjoined.

Following M.M. Dzhrbashjan [10], we introduce functions

$$\omega(\lambda) = u(1,\lambda) + u'(1,\lambda)\sin(\beta),$$

$$\widetilde{\omega(\lambda^*)} = z(0,\lambda^*) + z'(0,\lambda^*)\sin(\alpha)$$

where $u(x,\lambda)$ is a solution of the Cauchy problem

$$\begin{cases} u'' + a_m(x)u' + a_i(x)D_{0x}^{\alpha_i}\omega_i(x)u + \lambda u = 0, \\ u(0) = \sin(\alpha),\ u'(0) = -\cos(\alpha), \end{cases} \tag{3}$$

and $z(x,\lambda^*)$ is a solution of the problem

$$\begin{cases} z''(x) + a_m(x)z'(x) + \omega_i(x)D_{x1}^{\alpha_i}a_i(x)z + \lambda^* z = 0, \\ z(1) = \sin(\beta),\ z' = -\cos(\beta). \end{cases} \tag{4}$$

The existence and uniqueness of solutions of mutually adjoined problems (1)-(2) are already proved [3].

Clearly, a solution $u(x,\lambda)$ of the problem (3) is also a solution of the problem (1) iff

$$\omega(\lambda) = u(1,\lambda)\cos(\beta) + u'(1,\lambda)\sin(\beta).$$

There is a similar statement for the solutions of problems (4) and (2).

For construction of an orthogonal power series of eigenfunctions and associated functions of mutually adjoined problems by the method due to Dzhrbashjan M.M., it is required to use the following

Theorem 7.1. Let $a_m(0) = a_m(1) = 0$. The identity

$$(\lambda - \lambda^*)\int_0^1 u(x,\lambda)z(x,\lambda^*)dx = \omega(\lambda) - \widetilde{\omega(\lambda^*)} \tag{5}$$

holds for any parameters λ, λ^*.

Proof. Due to the definition of functions $u(x,\lambda)$ and $z(x,\lambda^*)$ as solutions of problems of Cauchy type, we have (3) and

$$\begin{cases} z''(x,\lambda^*)[a_m z(x,\lambda'')]' + \omega_i(x)D_{0x}^{\alpha_i}a_i(x) + \lambda^* z = 0, \\ z(1,\lambda^*) = \sin\beta,\ z'(1,\lambda^*) = -\cos\beta. \end{cases}$$

Multiplying both parts of the first equation of (1) by $z(x,\lambda^*)$ and integrating it from 0 to 1, we have

$$\begin{aligned}
&\int\limits_0^x z(x,\lambda^*)u''(x,\lambda)dx + \int\limits_0^1 a_m(x)z(x,\lambda^*)u'(x,\lambda)dx + \int\limits_0^x a_i(x)z(x,\lambda^*)D_{0x}^{\alpha_i}u(x,\lambda)dx \\
&+ \int\limits_0^x \lambda u(x,\lambda) * z(x,\lambda^*)dx \\
&= \omega(\lambda) - \widetilde{\omega}(\lambda^*)\int\limits_0^x u(x,\lambda)z''(x,\lambda)dx - \int\limits_0^1 [a_m(x)z(x,\lambda^*)]'u(x,\lambda) \\
&+ \int\limits_0^1 \omega_i(x)D_{x1}^{\alpha_i}a_i(t) * z(x,\lambda^*)u(x,\lambda)dx + \lambda\int\limits_0^1 u(x,\lambda)z(x,\lambda^*)dx.
\end{aligned} \tag{6}$$

At the same time, we also multiply both parts of the first equation of (2) by $u(x,\lambda)$ and integrate it from 0 up to 1, then we have

$$\begin{aligned}
&\int\limits_0^1 z(x,\lambda^*)u(x,\lambda)dx - \int\limits_0^1 [a_m(x)z(x,\lambda^*)]'u(x,\lambda)dx \\
&+ \int\limits_0^1 \omega_i(x)D_{x1}^{\alpha_i}a_i(t)z(x,\lambda^*)u(x,\lambda)dx + \lambda\int\limits_0^1 u(x,\lambda)z(x,\lambda^*)dx = 0.
\end{aligned} \tag{7}$$

Subtracting (7) from (6), we shall have

$$\int_0^1 z(x,\lambda^*)dx = \frac{\widetilde{\omega}(\lambda^*) - \omega(\lambda)}{\lambda - \lambda^*}.$$

Then we receive analogue of identity of Dzhrbashjan M.M.

Corollary 7.2 The formula

$$\omega(\lambda) = \widetilde{\omega}(\lambda) \tag{8}$$

holds, if $\lambda = \lambda^*$.

The parity of (8) follows from the identity (5).

We needed to construct biorthogonal system of eigenfunctions and attached functions of mutually adjoined problems (1).

Let $\{\lambda_n\}$ be a sequence of all eigenvalues of problem (1) arranged in the non-decreasing order of modulus

$$0 \le |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_n| \le \cdots$$

Note that, a same eigenvalue may appear repeatedly in the above sequence.

According to M.M. Dzhrbashjan [1], for each natural number $n \ge 1$, we designate p_n as a finite frequency rate of occurrence of number λ_n in sequence $\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|, \cdots\}$

Let

$$\{u_n(x)\}_{n=1}^{\infty}, \quad \{z_n(x)\}_{n=1}^{\infty}$$

be sequences of eigenfunctions of a problem (1) and (2) accordingly. As in [7] we prove that system of functions

$$\{u_n(x)\}_{n=1}^{\infty}, \quad \{u_n^*(x)\}_{n=1}^{\infty}$$

and

$$\{z_n(x)\}_{n=1}^{\infty}, \quad \{z_n^*(x)\}_{n=1}^{\infty}$$

are continuous on $(0, 1]$.

We shall call system of functions $\{u_n(x)\}_{n=1}^{\infty}$ as normal system of eigenfunctions and associated functions of problem (1), and system of functions $\{z_n(x)\}_{n=1}^{\infty}$ as normal system of eigenfunctions and associated functions of problem (2).

Then, we have the following important theorem of biorthogonality of these constructed systems.

Theorem 7.3. System of functions

$$\{u_n(x),\ z_n(x)\}$$

is biorthogonal on $[0, 1]$, i.e.

$$\int\limits_0^1 u_n(x) z_n(x) dx = \int\limits_0^1 z_n(x) u_n(x) dx = \begin{cases} \delta_{mn} 1, \ n \neq m, \\ 0, \ n = m. \end{cases}$$

The proof of this theorem is bulky and nothing different from the proof of corresponding theorem of M.M. Dzhrbashjan [1] if there is an identity

$$\int\limits_0^1 u_n(x) z_n(x) dx = \frac{\omega(\lambda) - \widetilde{\omega}(\lambda^*)}{\lambda - \lambda^*}$$

Systems of eigenfunctions and associated functions (3.19)-(3.22) are under construction only on the basis of identity (3.23) not coming back to a problem (3.7)-(3.8). Naturally from identity it is possible to obtain the sufficient information on a spectrum of a problem (3.7)-(3.8).

Chapter 3. Solving Two-Point Boundary Value Problems of Fractional Differential Equations(FBVPs)

In this chapter, the existence and uniqueness of solutions, and numerical methods for FBVPs with Caputo's derivatives [8] and with Riemann-Liouville derivatives [33] are studied, respectively.

FBVPs with Caputo's derivatives:

$$
\begin{aligned}
&{}_a^C D_t^\gamma y(t) + f\left(t, y(t)\right) = 0,\ a < t < b,\ 1 < \gamma \le 2, && (1)\\
&y(a) = \alpha, \quad y(b) = \beta && (2)
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_a^C D_t^\gamma y(t) + g\left(t,\ y(t),\ {}_a^C D_t^\theta y(t)\right) = 0,\ a < t < b, && (3)\\
&y(a) = \alpha,\ y(b) = \beta,\ 1 < \gamma \le 2,\ 0 < \theta \le 1, && (4)
\end{aligned}
$$

where $y : [a,b] \mapsto \mathbb{R}$, $f : [a,b] \times \mathbb{R} \mapsto \mathbb{R}$, $g : [a,b] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are continuous and satisfy Lipschitz conditions

$$
\begin{aligned}
|f(t,x) - f(t,y)| &\le K_f |x - y|, && (5)\\
|g(t,x_2,y_2) - g(t,x_1,y_1)| &\le K_g |x_2 - x_1| + L_g |y_2 - y_1| && (6)
\end{aligned}
$$

with Lipschitz constants $K_f,\ K_g,\ L_g > 0$.
FBVPs with Riemann-Liouville derivatives [33]:

$$
\begin{aligned}
&{}_a^{RL} D_t^\gamma y(t) + f\left(t, y(t)\right) = 0,\ a < t < b,\ 1 < \gamma \le 2, && (7)\\
&y(a) = 0,\ y(b) = \beta, && (8)
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_a^{RL} D_t^\gamma y(t) + g\left(t,\ y(t),\ {}_a^{RL} D_t^\theta y(t)\right) = 0,\ a < t < b, && (9)\\
&y(a) = 0,\ y(b) = \beta,\ 1 < \gamma \le 2,\ 0 < \theta \le \gamma - 1, && (10)
\end{aligned}
$$

where $y : [a,b] \mapsto \mathbb{R}$, $f : [a,b] \times \mathbb{R} \mapsto \mathbb{R}$, $g : [a,b] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are continuous and satisfy Lipschitz conditions (5) and (6), respectively.

1. The basic concepts

In order to state the problems in concern we introduce the definitions and properties of Caputo's derivatives, and Riemann-Liouville fractional integrals in [8] and [33].

Definition 1.1 (see [29]) *Provided $\gamma > 0$, $n - 1 < \gamma \le n$ and let $\mathbb{C}^n[a,b] := \{y(t) : [a,b] \to \mathbb{R}; y(t)$ has a continuous n-th derivative$\}$.*

(1) The operator ${}_a^{RL} D_t^\gamma$ defined by

$$
{}_a^{RL} D_t^\gamma y(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-\tau)^{n-\gamma-1} y(\tau) d\tau \qquad (11)
$$

for $t \in [a,b]$ and $y(t) \in \mathbb{C}^n[a,b]$, is called the Riemann-Liouville differential operator of order γ.

(2) The operator ${}_{a}^{C}D_{t}^{\gamma}$ *defined by*

$$ {}_{a}^{C}D_{t}^{\gamma}y(t) = \frac{1}{\Gamma(n-\gamma)}\int_{a}^{t}(t-\tau)^{n-\gamma-1}\left(\frac{d}{d\tau}\right)^{n}y(\tau)d\tau \tag{12} $$

for $t \in [a,b]$ *and* $y(t) \in \mathbb{C}^{n}[a,b]$, *is called the Caputo differential operator of order* γ.

Definition 1.2 (see [29])*Provided* $\gamma > 0$, *the operator* J_{a}^{γ}, *defined on* $\mathbb{L}_1[a,b]$ *by*

$$ J_{a}^{\gamma}y(t) = \frac{1}{\Gamma(\gamma)}\int_{a}^{t}(t-\tau)^{\gamma-1}y(\tau)d\tau \tag{13} $$

for $t \in [a,b]$, *is called the Riemann-Liouville fractional integral operator of order* γ, *where* $\mathbb{L}_1[a,b] := \{y(t) : [a,b] \to \mathbb{R}; y(t)$ *is measurable on* $[a,b]$ *and* $\int_a^b |y(t)|dt < \infty\}$.

Lemma 1.3 (see [31])

(**1**) *Let* $\gamma > 0$. *Then, for every* $f \in L_1[a,b]$,

$$ {}_{a}^{RL}D_{t}^{\gamma}J_{a}^{\gamma}f = f \tag{14} $$

almost everywhere.

(**2**) *Let* $\gamma > 0$ *and* $n-1 < \gamma \leq n$. *Assume that* f *is such that* $J_{a}^{n-\gamma}f \in \mathbb{A}^{n}[a,b]$. *Then,*

$$ J_{a}^{\gamma}\,{}_{a}^{RL}D_{t}^{\gamma}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{\gamma-k-1}}{\Gamma(\gamma-k)}\lim_{z\to a^{+}}\frac{d^{n-k-1}}{dz^{n-k-1}}J_{a}^{n-\gamma}f(z). \tag{15} $$

(**3**) *Let* $\gamma > \theta > 0$ *and* f *be continuous. Then*

$$ {}_{a}^{C}D_{t}^{\gamma}J_{a}^{\gamma}f = f, \quad {}_{a}^{C}D_{t}^{\theta}J_{a}^{\gamma}f = J_{a}^{\gamma-\theta}f. \tag{16} $$

(**4**) *Let* $\gamma \geq 0$, $n-1 < \gamma \leq n$ *and* $f \in \mathbb{A}^{n}[a,b]$. *Then*

$$ J_{a}^{\gamma}\,{}_{a}^{C}D_{t}^{\gamma}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{D^{k}f(a)}{k!}(t-a)^{k}, \tag{17} $$

where $\mathbb{A}^{n}[a,b]$ *is a set of functions with absolutely continuous derivative of order* $n-1$.

Let $\mathbb{S}$ be a Banach space, $\mathcal{T} : \mathbb{S} \longmapsto \mathbb{S}$ be a mapping, and $\|\cdot\|$ denote the norm of $\mathbb{S}$.

Definition 1.4 (see [25]) *If there exists a constant* ρ *(*$0 \leq \rho < 1$*), such that*

$$ \|\mathcal{T}x - \mathcal{T}y\| \leq \rho\|x-y\| \tag{18} $$

for any $x, y \in \mathbb{S}$*, then* $\mathcal{T}$ *is said to be a* contractive mapping *of* $\mathbb{S}$.

Lemma 1.5 (**Contractive Mapping Principle**) (see [25]) *If* $\mathcal{T}$ *is a contractive mapping of Banach space* $\mathbb{S}$*, then there exists a unique fixed point* $y \in \mathbb{S}$ *satisfying* $y = \mathcal{T}y$*.*

2. Existence and uniqueness of the solutions for FBVPs

In this section, the existence and uniqueness of the solutions for FBVPs (7-8), (9-10), (1-2) and (3-4) are studied. Without loss of generality, only the case of homogeneous boundary conditions $\alpha = \beta = 0$ in the above four kinds of FBVPs are considered.

Lemma 2.1 (1) FBVP (1-2) is equivalent to

$$y(t) = \int_a^b G(t,s) f(s, y(s)) ds, \tag{19}$$

where $G(t,s)$ is called the fractional Green function defined as follows:

$$G(t,s) = \begin{cases} \dfrac{(t-a)(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)} - \dfrac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}, & a \le s \le t \le b, \\ \dfrac{(t-a)(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)}, & a \le t \le s \le b. \end{cases} \tag{20}$$

(2) (see [33]) FBVP (7-8) is equivalent to

$$y(t) = \int_a^b \widehat{G}(t,s) f(s, y(s)) ds, \tag{21}$$

where $\widehat{G}(t,s)$ is the fractional Green function defined as follows:

$$\widehat{G}(t,s) = \begin{cases} \dfrac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \left(\dfrac{b-s}{b-a}\right)^{\gamma-1} - \dfrac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}, & a \le s \le t \le b, \\ \dfrac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \left(\dfrac{b-s}{b-a}\right)^{\gamma-1}, & a \le t \le s \le b. \end{cases} \tag{22}$$

(3) FBVP (3-4) is equivalent to

$$y(t) = \int_a^b G(t,s) g\left(s,\ y(s),\ {}_a^C D_s^\theta y(s)\right) ds, \tag{23}$$

where $G(t,s)$ is the fractional Green function defined in (20).

(4) (see [33]) FBVP (9-10) is equivalent to

$$y(t) = \int_a^b \widehat{G}(t,s) g\left(s,\ y(s),\ {}_a^{RL} D_s^\theta y(s)\right) ds, \tag{24}$$

where $\widehat{G}(t,s)$ is the fractional Green function defined in (22).

Proof: We only give the proof for (1). The case for (3) is similar to that for (1). The proof of (2) and (4) are referred to [33].

According to **Lemma 1.3(4)**, acting the operator J_a^γ to both sides of the equation in (1) yields

$$y(t) - y(a) - y'(a)(t-a) + J_a^\gamma f(t, y(t)) = 0. \tag{25}$$

Since $y(a) = y(b) = 0$, from (25) one easily obtains

$$y'(a) = \int_a^b \frac{(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)} f(s, y(s)) ds, \tag{26}$$

and then

$$\begin{aligned} y(t) &= y'(a)(t-a) - J_a^\gamma f(t, y(t)) \\ &= \int_a^b \frac{(t-a)(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)} f(s, y(s)) ds - \int_a^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} f(s, y(s)) ds \\ &= \int_a^b G(t,s) f(s, y(s)) ds. \end{aligned}$$

Conversely, acting the operator ${}_a^C D_t^\gamma$ to both sides of (19) yields

$$\begin{aligned} {}_a^C D_t^\gamma y(t) &= {}_a^C D_t^\gamma \int_a^b G(t,s) f(s, y(s)) ds \\ &= \int_a^b \frac{(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)} f(s, y(s)) ds \, {}_a^C D_t^\gamma (t-a) - {}_a^C D_t^\gamma J_a^\gamma f(t, y(t)) \\ &= -f(t, y(t)), \end{aligned}$$

and the homogeneous boundary condition is verified easily. $\square$

Let

$$\begin{aligned} &\mathbb{P} = \mathbb{C}^0[a,b], \\ &\mathbb{P}_1 = \mathbb{C}^1[a,b] := \left\{ y(t) |\ y(t), y'(t) \in \mathbb{C}^0[a,b] \right\}, \\ &\mathbb{P}_2 = \mathbb{C}^\theta[a,b] := \left\{ y(t) |\ y(t) \in \mathbb{C}^0[a,b],\ {}_a^{RL} D_t^\theta y(t) \in \mathbb{C}^0[a,b] \right\}. \end{aligned}$$

where $\mathbb{C}^0[a,b]$ denote the space of all continuous functions on $[a,b]$. We define the norm $\|\cdot\|_f$, $\|\cdot\|_g$, $\|\cdot\|_{\hat{g}}$ and the operator $\mathcal{T}_f$, $\widehat{\mathcal{T}}_f$, $\mathcal{T}_g$, $\widehat{\mathcal{T}}_g$ as follows:

$$\begin{aligned} &\|y\|_f := K_f \max_{a \le t \le b} |y(t)|, \\ &\mathcal{T}_f y(t) := \int_a^b G(t,s) f(s, y(s)) ds, \\ &\widehat{\mathcal{T}}_f y(t) := \int_a^b \widehat{G}(t,s) f(s, y(s)) ds, \quad \forall\ y(t) \in \mathbb{P}. \end{aligned}$$

$$\|y\|_g := \max_{a\le t\le b} \left[\ K_g|y(t)| + L_g \left|{}_a^C D_t^\theta y(t)\right|\ \right],$$

$$\mathcal{T}_g y(t) := \int_a^b G(t,s) g\left(s,\ y(s),\ {}_a^C D_s^\theta y(s)\right) ds, \quad \forall\ y(t) \in \mathbb{P}_1.$$

$$\|y\|_{\hat{g}} := \max_{a\le t\le b} \left[\ K_g|y(t)| + L_g \left|{}_a^{RL} D_t^\theta y(t)\right|\ \right],$$

$$\widehat{\mathcal{T}}_g y(t) := \int_a^b \widehat{G}(t,s) g\left(s,\ y(s),\ {}_a^{RL} D_s^\theta y(s)\right) ds, \quad \forall\ y(t) \in \mathbb{P}_2.$$

Obviously, $\mathbb{P}$, $\mathbb{P}_1$, $\mathbb{P}_2$ are complete norm space with respect to $\|\cdot\|_f$, $\|\cdot\|_g$ and $\|\cdot\|_{\hat{g}}$, respectively. The operators $\mathcal{T}_f$, $\widehat{\mathcal{T}}_f$, $\mathcal{T}_g$, $\widehat{\mathcal{T}}_g$ are continuous. Now (19) and (21) can be rewritten as $y = \mathcal{T}y$ and $y = \widehat{\mathcal{T}}_f y$, for $y \in \mathbb{P}$; (23), (24) can be rewritten as $y = \mathcal{T}_g y$ and $y = \widehat{\mathcal{T}}_g y$, respectively. According to the contractive mapping principle, "finding a sufficient condition for existence and uniqueness of the solution for FBVP (1-2) (or (7-8) or (3-4) or (9-10))" is equivalent to "finding a sufficient condition under which $\mathcal{T}_f$ (or $\widehat{\mathcal{T}}_f$ or $\mathcal{T}$ or $\widehat{\mathcal{T}}_g$) is a contractive mapping of $\mathbb{P}$(or $\mathbb{P}_1$ or $\mathbb{P}_2$)". Then, the following theorems are given.

Theorem 2.2 (1) Let f be a continuous function on $[a,b] \times \mathbb{R}$ and satisfy the Lipschitz condition (5).

(1.1) If

$$\frac{2K_f(b-a)^\gamma}{\Gamma(\gamma+1)} < 1, \tag{27}$$

then there exists a unique solution for FBVP (1-2) in $\mathbb{P}$.

(1.2) (see [33]) If

$$K_f \frac{(\gamma-1)^{\gamma-1}(b-a)^\gamma}{\gamma^\gamma \Gamma(\gamma+1)} < 1, \tag{28}$$

then there exists a unique solution for FBVP (7-8) in the space $\mathbb{P}$.

(2) Let g be a continuous function on $[a,b] \times \mathbb{R} \times \mathbb{R}$ and satisfy the Lipschitz condition (6).

(2.1) If

$$K_g \frac{2(b-a)^\gamma}{\Gamma(\gamma+1)} + L_g \left[\frac{(b-a)^{\gamma-\theta}}{\Gamma(\gamma+1)\Gamma(2-\theta)} + \frac{(b-a)^{\gamma-\theta}}{\Gamma(\gamma+1-\theta)}\right] < 1, \tag{29}$$

then there exists a unique solution for FBVP (3-4) in $\mathbb{P}_1$.

(2.2) (see [33]) If

$$K_g\frac{(\gamma-1)^{\gamma-1}(b-a)^{\gamma}}{\gamma^{\gamma}\Gamma(\gamma+1)}+L_g\frac{(2\gamma-\theta)(b-a)^{\gamma-\theta}}{\gamma\Gamma(\gamma-\theta+1)}<1, \qquad (30)$$

then there exists a unique solution for FBVP (9-10) in the space $\mathbb{P}_2$.

Proof:

The proof for (1.1): For $u(t)$, $v(t)\in\mathbb{P}$, and $(t,s)\in[a,b]\times[a,b]$, according to the definition for the operator "$\mathcal{T}_f$" we have

$$\begin{aligned}|\mathcal{T}_f u(t)-\mathcal{T}_f v(t)| &\le \int_a^b |G(t,s)|\cdot|f(s,u(s))-f(s,v(s))|ds\\ &\le \|u-v\|_f\left\{\int_a^b\frac{(t-a)(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)}ds+\int_a^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}ds\right\}\\ &\le \frac{2(b-a)^{\gamma}}{\Gamma(\gamma+1)}\|u-v\|_f.\end{aligned}$$

Thus

$$\|\mathcal{T}_f u-\mathcal{T}_f v\|_f=\max_{a\le t\le b}K_f|\mathcal{T}_f u(t)-\mathcal{T}_f v(t)|\le\frac{2K_f(b-a)^{\gamma}}{\Gamma(\gamma+1)}\|u-v\|_f.$$

Considering (27), we finish the proof according to **Lemma 1.5**.

The proof for (2.1): On one hand, we have

$$|\mathcal{T}_g u(t)-\mathcal{T}_g v(t)| \le \frac{2(b-a)^{\gamma}}{\Gamma(\gamma+1)}\|u-v\|_g$$

for $u(t)$, $v(t)\in\mathbb{P}_1$, and $(t,s)\in[a,b]\times[a,b]$, which is similar to (1.1). On the other hand, according to **Lemma 1.3**, we have

$${}_a^C D_t^{\theta}\mathcal{T}_g u(t)=\frac{(t-a)^{1-\theta}}{\Gamma(2-\theta)}\int_a^b\frac{(b-s)^{\gamma-1}}{\Gamma(\gamma)(b-a)}g\left(s,\ u(s),\ {}_a^C D_s^{\theta}u(s)\right)ds-J_a^{\gamma-\theta}g\left(t,\ u(t),\ {}_a^C D_t^{\theta}u(t)\right).$$

And then

$$\left|{}_a^C D_t^{\theta}\mathcal{T}_g u-{}_a^C D_t^{\theta}\mathcal{T}_g v\right| \le \|u-v\|_g\cdot\left[\frac{(b-a)^{\gamma-\theta}}{\Gamma(2-\theta)\Gamma(\gamma+1)}+\frac{(b-a)^{\gamma-\theta}}{\Gamma(\gamma-\theta+1)}\right].$$

Combined with the definition of $\|\cdot\|_g$ and **Lemma 1.5**, (29) holds.

The proof for (1.2) and (2.2) are referred to [33].

3. Shooting methods for FBVPs

In this section, single shooting methods are applied to solve the FBVPs (1-2), (3-4), (7-8) and (9-10) numerically.

According to the idea of the shooting method, a FBVP is turned into a fractional initial value problem (FIVP) which can be solved by some suitable numerical method (see [26]-[27], [28], [31]). We write down the corresponding procedure for FBVPs.

Denote the corresponding initial value conditions of FBVPs (1-2) and (3-4) as

$$y(a) = a_0, \quad y'(a) = a_1, \quad a_0, a_1 \in \mathbb{R}. \tag{31}$$

Then FBVPs (1-2) and (3-4) can turn into FIVPs (1, 31) and (3, 31), respectively.

Usually, not all a_k $(k = 0, 1)$ are equal to zero, in other words, the initial value conditions (31) are inhomogeneous. Setting

$$z(t) = y(t) - a_0 - a_1(t - a), \tag{32}$$

(1, 31) and (3, 31) can be transformed into another FIVPs with homogeneous initial value conditions.

For a given equispaced mesh $a = t_0 < t_1 < \cdots < t_N = b$ with stepsize $h = (b-a)/N$, we give a fractional backward difference scheme of order one (refer to [31]) to solve FIVPs (1) and (3) with homogeneous initial value conditions $y(a) = 0, \;\; y'(a) = 0$:

$$z_m \;=\; -h^{\gamma} f(t_m, z_m) - \sum_{k=1}^{m} \omega_k z_{m-k}, \tag{33}$$

and

$$z_m \;=\; -h^{\gamma} g(t_m, z_m, \frac{1}{h^{\theta}} \sum_{i=0}^{m} \tilde{\omega}_i z_{m-i}) - \sum_{k=1}^{m} \omega_k z_{m-k}, \tag{34}$$

where

$$\omega_0 \;=\; 1, \;\; \omega_k \;=\; (1 - \frac{\gamma+1}{k})\omega_{k-1}, \;\; k = 1, 2, \cdots, N, \tag{35}$$

$$\tilde{\omega}_0 \;=\; 1, \;\; \tilde{\omega}_i \;=\; (1 - \frac{\theta+1}{i})\tilde{\omega}_{i-1}, \;\; i = 1, 2, \cdots, N. \tag{36}$$

The case for (7-8) and (9-10) is referred to [33]. The reader should note that the initial values take the following form

$${}_{a}^{RL}D_t^{\gamma-1} y(a) = b_1 \in \mathbb{R}, \qquad \lim_{t \to a^+} J_a^{2-\gamma} y(t) = b_2 \in \mathbb{R}. \tag{37}$$

It is easy to check that

$$b_2 = \lim_{t \to a^+} J_a^{2-\gamma} y(t) = 0. \tag{38}$$

Usually, $b_1 \neq 0$ and let

$$z(t) = y(t) - \frac{b_1(t-a)^{\gamma-1}}{\Gamma(\gamma)}. \tag{39}$$

in (7, 37) and (9, 37). Then they are transformed into FIVPs with homogeneous initial conditions. (33) or (34) is also employed to simulate them.

3.1. Shooting method for linear problems

The linear case of fractional two-point boundary value problem (1) with homogeneous boundary value conditions

$$\begin{aligned} &{}_a^C D_t^\gamma y(t) + c(t)y(t) + d(t) = 0,\ a < t < b,\ 1 < \gamma \leq 2, && (40)\\ &y(a) = 0, \quad y(b) = 0 && (41) \end{aligned}$$

where $c(t),\ d(t) \in \mathbb{C}^0[a,b]$.

According to **Theorem 2.2**, if $b-a < \left[\frac{\Gamma(\gamma+1)}{2K_f}\right]^{\frac{1}{\gamma}}$ with $K_f = \max\limits_{a \leq t \leq b} |c(t)|$, then there exists a unique solution for (40-41). In order to apply the shooting method, we choose an initial value ξ_1, ξ_2 for y' (denote initial value of exact solution $y'(a) = \xi^*$), respectively:

$$\begin{aligned} y(a) = 0, \quad y'(a) = \xi_1. && (42)\\ y(a) = 0, \quad y'(a) = \xi_2. && (43) \end{aligned}$$

Then FIVP (40, 42) and FIVP (40, 43) have unique solution, denoted by $y(t;\xi_1)$, $y(t;\xi_2)$. Usually, $\xi_1 \neq \xi_2 \neq \xi^*$, $y(b;\xi_1) \neq y(b;\xi_2) \neq 0$, thus $y(t;\xi_1) \neq y$ and $y(t;\xi_2) \neq y$. Suppose that $y(b;\xi_1) \neq y(b;\xi_2)$, let $\lambda := \dfrac{y(b) - y(b;\xi_2)}{y(b;\xi_1) - y(b;\xi_2)}$ and

$$y(t) := \lambda y(t;\xi_1) + (1-\lambda) y(t;\xi_2). \tag{44}$$

It is easy to show that (44) is the solution of FBVP (40-41).

In [8], the error estimates for linear case are also considered.

For a given equispaced mesh $a = t_0 < t_1 < \cdots < t_N = b$ with stepsize $h = (b-a)/N$, we denote numerical solution of FIVPs (40, 42) and FIVPs (40, 43) as $\{y_n(\xi_1)\}_{n=0}^N$ and $\{y_n(\xi_2)\}_{n=0}^N$, respectively. And then, $\hat{\lambda} = \dfrac{y(b) - y_N(\xi_2)}{y_N(\xi_1) - y_N(\xi_2)}$ and the numerical solution

$$y_n = \hat{\lambda} y_n(\xi_1) + (1-\hat{\lambda}) y_n(\xi_2), \quad n = 0, 1, 2, \cdots, N. \tag{45}$$

Generally, we suppose that the scheme solving FIVPs (40, 42) have convergent order $\mathcal{O}(h^r)$, i.e.,

$$\begin{cases} \max_{0\le n\le N} |y_n(\xi_1) - y(t_n;\xi_1)| = \mathcal{O}(h^r), \\ \max_{0\le n\le N} |y_n(\xi_2) - y(t_n;\xi_2)| = \mathcal{O}(h^r). \end{cases} \tag{46}$$

Due to (44-45), we consider $\{|y(t_n) - y_n|\}_{n=0}^{N}$, then we get

$$\max_{0\le n\le N} |y(t_n) - y_n| = \mathcal{O}(h^r). \tag{47}$$

When come to the linear case of (7-8), we only need to replace (42) and (43) with initial value conditions

$$ {}_{a}^{RL}D_t^{\gamma-1} y(t)\Big|_{t=a} = \xi_1, \quad \lim_{t\to a^+} J_a^{2-\gamma} y(t) = 0, \tag{48}$$

$$ {}_{a}^{RL}D_t^{\gamma-1} y(t)\Big|_{t=a} = \xi_2, \quad \lim_{t\to a^+} J_a^{2-\gamma} y(t) = 0. \tag{49}$$

The details are referred to [33]. Again, the shooting procedure for linear FBVPs (3-4) and (9-10) are similar to that of (1-2) and (7-8), respectively.

3.2. Shooting method for nonlinear problems

For nonlinear FBVP (1) with homogeneous boundary value conditions (41), we consider the following FIVP :

$$ {}_{a}^{C}D_t^{\gamma} y(t) + f(t, y(t)) = 0, \ a \le t \le b, \ 1 < \gamma \le 2, \tag{50}$$

$$ y(a) = 0, \ y'(a) = \xi, \tag{51}$$

The FIVPs (50) is corresponding to FBVP (1, 41) with analytic solutions denoted as $y(t;\xi)$. In general, $y(b;\xi) \neq 0$. Once a zero point ξ^* of $\phi(\xi) := y(b;\xi)$ is found, one obtains $y(t;\xi^*)$, the solution of FBVP (1, 41). When $y(t;\xi)$, and hence $\phi(\xi)$ are continuously differentiable with respect to ξ, Newton's method can be employed to determine ξ^*. Starting with an initial approximation $\xi^{(0)}$, one gets $\xi^{(k)}$ as follows:

$$\xi^{(k+1)} = \xi^{(k)} - \frac{\phi(\xi^{(k)})}{\phi'(\xi^{(k)})}. \tag{52}$$

On one hand, $y(b;\xi)$, hence $\phi(\xi)$ can be determined by solving FIVP (50) numerically. On the other hand, it is easy to check that $w(t;\xi) := \dfrac{\partial y(t;\xi)}{\partial \xi}$ is the solution of the following FIVP (refer to [30]):

$$ {}_{a}^{C}D_t^{\gamma} w(t;\xi) + \frac{\partial f(t, y(t;\xi))}{\partial y} w(t;\xi) = 0, \tag{53}$$

$$ w(a;\xi) = 0, \ w'(a;\xi) = 1, \ a \le t \le b, \ 1 < \gamma \le 2. \tag{54}$$

So we can compute $w(b;\xi)$, i.e., $\phi'(\xi)$ by solving FIVPs (53) numerically. However, considering of the computation complexity for $\dfrac{\partial f(t,y(t;\xi))}{\partial y}$ in practice, we usually calculate the difference quotient

$$\triangle\phi(\xi^{(k)}) := \frac{\phi(\xi^{(k)}+\triangle\xi^{(k)})-\phi(\xi^{(k)})}{\triangle\xi^{(k)}}$$

instead of the derivative $\phi'(\xi^{(k)})$ itself, where $\triangle\xi^{(k)}$ is a sufficiently small number. And the formula (52) is replaced by

$$\xi^{(k+1)} = \xi^{(k)} - \frac{\phi(\xi^{(k)})}{\triangle\phi(\xi^{(k)})}. \tag{55}$$

In short, the procedure of solving FBVP (1, 41) by shooting method is displayed as follows:
step 1. choose a starting value $\xi^{(0)}$ and an iterative precision ϵ for Newton's method;
then for k=0,1,2,$\cdots$,
step 2. obtain $y(b;\xi^{(k)})$ by solving FIVP (50) with $y'(a)=\xi^{(k)}$ and compute $\phi(\xi^{(k)})$;
step 3. choose a sufficiently small number $\triangle\xi^{(k)}\neq 0$ and determine $y(b;\xi^{(k)}+\triangle\xi^{(k)})$ by solving the FIVP (50) with $y'(a)=\xi^{(k)}+\triangle\xi^{(k)}$, and then compute $\phi(\xi^{(k)}+\triangle\xi^{(k)})$;
step 4. compute $\triangle\phi(\xi^{(k)})$ and determine $\xi^{(k+1)}$ by formula (55);
repeat *steps 2-4* until $|\xi^{(k+1)}-\xi^{(k)}|\leq\epsilon$ and denote the final $\xi^{(k+1)}$ as ξ^*;
step 5. finally, we obtain the numerical solution of FBVP (1, 41) by solving FIVP (50) with $y'(a)=\xi^*$.

Next, we want to give the error analysis of shooting method for nonlinear FBVP (1). The error consists of two parts. One is from the error between $|\xi^*-\tilde{\xi}|$. By shooting method, we can only get the approximation $\tilde{\xi}$ but exact initial value $\xi^*(=y'(a))$. That is to say, we solve FIVP

$$\begin{cases} {}_{\mathbf{a}}^{\mathbf{C}}\mathbf{D}_{\mathbf{t}}^{\alpha}y(t)+f(t,y(t))=0,\\ \quad y(a)=0,\quad y'(a)=\tilde{\xi} \end{cases} \tag{56}$$

instead of FIVP

$$\begin{cases} {}_{\mathbf{a}}^{\mathbf{C}}\mathbf{D}_{\mathbf{t}}^{\alpha}y(t)+f(t,y(t))=0,\\ \quad y(a)=0,\quad y'(a)=\xi^*. \end{cases} \tag{57}$$

which is an initial perturbation problem in fact. Denote the exact solutions of (56) and (57) as $y(t;\tilde{\xi})$ and $y(t;\xi^*)$, respectively.

Lemma 3.1 (see [32]) Suppose $\alpha > 0$, $A(t)$ is a nonnegative function locally integrable on $[a, b]$ and $B(t)$ is a nonnegative, nondecreasing continuous function defined on $[a, b]$, and suppose $\psi(t)$ is nonnegative and locally integrable on $[a, b]$ with

$$\psi(t) \leq A(t) + B(t) \int_a^t (t-\tau)^{\alpha-1} \psi(\tau) d\tau, \quad t \in [a, b]. \tag{58}$$

on this interval. Then

$$\psi(t) \leq A(t) E_\alpha \left(B(t) \Gamma(\alpha) (t-a)^\alpha \right), \tag{59}$$

where E_α is Mittag-Leffler function defined by

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)}. \tag{60}$$

According to the above lemma, We get the error estimation.

$$|y(t;\xi^*) - y(t;\tilde{\xi})| \leq |\xi^* - \tilde{\xi}|(t-a) E_\alpha \left(K_f (t-a)^\alpha \right) \leq C_1 |\xi^* - \tilde{\xi}| \quad, t \in [a, b] \tag{61}$$

where $C_1 := \max_{t \in [a,b]} (t-a) E_\alpha \left(K_f (t-a)^\alpha \right)$.

Another part of error is from numerical solving procedure of FIVP (50) by scheme (33, 35). Suppose that the scheme of FIVP converges with order r, and denote the approximate solution of $y(t_n; \tilde{\xi})$ as $y_n(\tilde{\xi})$, we get $\exists C_2 > 0$, s.t.

$$\max_{0 \leq n \leq N} |y(t_n; \tilde{\xi}) - y_n(\tilde{\xi})| \leq C_2 h^r. \tag{62}$$

To sum up, we have

$$\begin{aligned} |y(t_n;\xi^*) - y_n(\tilde{\xi})| &= |y(t_n;\xi^*) - y(t_n;\tilde{\xi}) + y(t_n;\tilde{\xi}) - y_n(\tilde{\xi})| \\ &\leq |y(t_n;\xi^*) - y(t_n;\tilde{\xi})| + |y(t_n;\tilde{\xi}) - y_n(\tilde{\xi})| \end{aligned}$$

that is,

$$\max_{0 \leq n \leq N} |y(t_n;\xi^*) - y_n(\tilde{\xi})| \leq C_1 |\xi^* - \tilde{\xi}| + C_2 h^r. \tag{63}$$

When come to (7, 41), we turn it into its corresponding FIVP

$${}_a^{RL}D_t^\gamma y(t) + f\left(t, y(t)\right) = 0, \qquad a \leq t \leq b, \quad 1 < \gamma \leq 2 \tag{64}$$

$${}_a^{RL}D_t^{\gamma-1} y(t) \Big|_{t=a} = \xi, \quad \lim_{t \to a^+} J_a^{2-\gamma} y(t) = 0. \tag{65}$$

And the numerical procedure of simulating (7, 41) is completely similar to that of (1, 41). The reader can refer to [33] for details.

Again, the procedure of shooting method for (2) and (8) with homogenous boundary conditions (41) are similar to the case of (1, 41) and (7, 41), respectively.

4. Numerical experiments for solving FBVPs

In this section, two numerical examples are given to show the feasibility and validity of single shooting methods for the FBVPs.

First, let us introduce the parameters. $a = 0 = t_0 < t_1 < \cdots < t_n = b$ is a given equispaced mesh with stepsize $h = (b-a)/n$, n=100, $\gamma = 1.5$, $\theta = 0.5$. For the linear cases in examples 1, the two "initial speeds" $\xi_1 = 0.5$, $\xi_2 = 1.5$. For the nonlinear cases in examples 2, $\epsilon = 10^{-12}$, $\xi^{(0)} = 0$, $\triangle\xi^{(k)} \equiv 10^{-8}$.

Example 1 Consider the following linear FBVP

$$\begin{aligned}
&{}_0^C D_t^{1.5} y(t) + \frac{1}{3} y(t) + \frac{1}{4}\, {}_0^C D_t^{0.5} y(t) = r(t), \quad 0 < t < 1, \\
&y(0) = 0, \quad y(1) = 0,
\end{aligned}$$

where

$$r(t) = -\frac{7t^{0.5}}{2\sqrt{\pi}} + \frac{t}{3} - \frac{2t^{1.5}}{3\sqrt{\pi}} - \frac{t^2}{3}.$$

Since

$$\begin{aligned}
&|g(t,u_2,v_2) - g(t,u_1,v_1)| \le \frac{1}{3}|u_2 - u_1| + \frac{1}{4}|v_2 - v_1|, \\
&K_g = \frac{1}{3},\ L_g = \frac{1}{4},\ a = 0,\ b = 1,\ \gamma = 1.5,\ \theta = 0.5, \\
&K_g \frac{2(b-a)^\gamma}{\Gamma(\gamma+1)} + L_g \left[\frac{(b-a)^{\gamma-\theta}}{\Gamma(2-\theta)\Gamma(\gamma+1)} + \frac{(b-a)^{\gamma-\theta}}{\Gamma(\gamma-\theta+1)} \right] \approx 0.96 < 1,
\end{aligned}$$

there exists a unique solution for this FBVP according to **Theorem 2.2**. In fact, one can easily check that $y(t) = t(1-t)$ is the analytical solution. The errors between the numerical solution (obtained by using shooting method mentioned above) and the analytical solution at mesh points are plotted in Figure 1. And the true solution (denoted by real line) and numerical solution (denoted by □) on equispaced mesh are plotted in Figure 2.

Example 2 Consider the following nonlinear FBVP

$$\begin{aligned}
{}_{-1}^{C} D_t^{1.5} y(t) &+ \sin(y) + r(t) = 0, \quad -1 < t < 1, \\
y(-1) &= 0, \quad y(1) = 0,
\end{aligned}$$

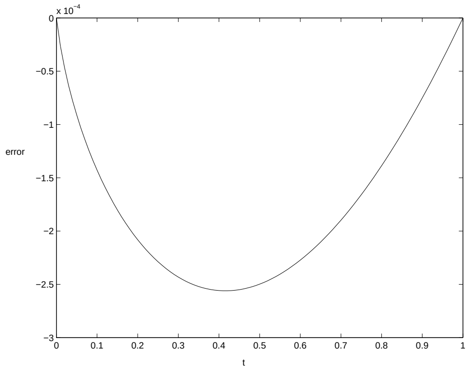

Figure 1: the shooting error for Example 1

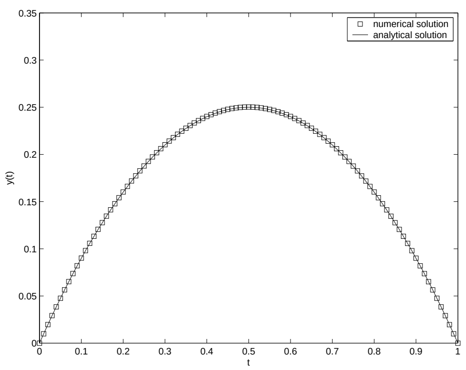

Figure 2: the true solution and numerical solution of Example 1

where

$$\begin{aligned} r(t) &= \sin(C_{t+1}(2,2\pi)) + C_{t+1}(0.5,2\pi). \\ C_t(\alpha,\omega) &= t^{\alpha}\sum_{j=0}^{\infty}\frac{(-1)^j(\omega t)^{2j}}{\Gamma(\alpha+2j+1)}. \end{aligned}$$

Since

$$\begin{aligned} & |f(t,u_2)-f(t,u_1)| = |\sin(u_2)-\sin(u_1)| \le |u_2-u_1|\,, \\ & K_f = 1,\ a=-1,\ b=1,\ \gamma=1.5, \\ & K_f\frac{(\gamma-1)^{\gamma-1}(b-a)^{\gamma}}{\gamma^{\gamma}\Gamma(\gamma+1)} \approx 0.82 < 1, \end{aligned}$$

there exists a unique solution for this FBVP according to **Theorem 2.2**. In fact, one can easily check that $y(t) = C_{t+1}(2,2\pi)$ is the analytical solution. The errors between the numerical solution (obtained by using shooting method mentioned above) and the analytical solution (approximated by summarizing the former 21 terms of infinite series $C_{t+1}(2,2\pi)$ because of $\frac{(2\pi)^{40}}{\Gamma(43)} \approx 6.02\times10^{-20}$) at different points are plotted in Figure 3. And the analytical solution (denoted by real line) and numerical solution (denoted by $\square$) on equispaced mesh are plotted in Figure 4.

The rates of convergence and maximum errors for example 2 between the numerical solution and the analytical solution are given in table 1. Table 1 show that the shooting method is order one, which is in good agreement with the fact that (33, 35) or (34, 36) is a method of order one.

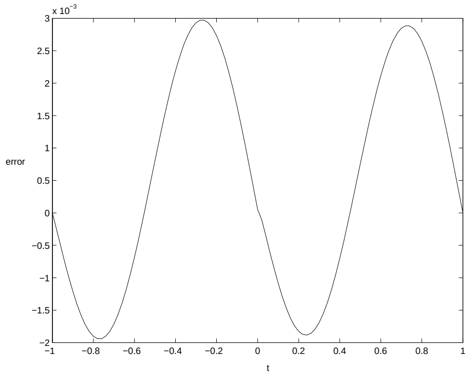

Figure 3: the shooting error for Example 2

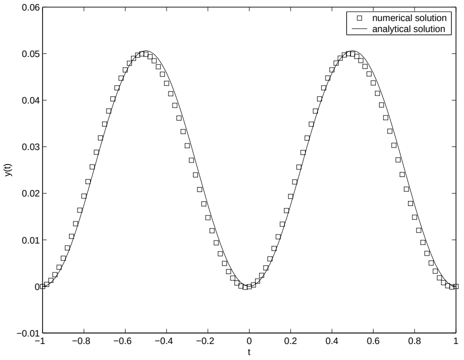

Figure 4: the true solution and numerical solution of Example 2

Table 1

stepsize $h = 1/n$	$\max_{1 \leq i \leq n} \lvert y(t_i) - y_n(t_i) \rvert$	rates of convergence
$1/16$	$2.46e - 2$	
$1/32$	$1.06e - 2$	1.2198
$1/64$	$4.9e - 3$	1.1013
$1/128$	$2.34e - 3$	1.0740
$1/256$	$1.31e - 3$	1.0471

The above numerical results indicates that the single shooting method is a successful tool to solve fractional boundary value problems (1-2), (3-4). The numerical experiments for FBVPs (7-8) and (9-10) can be referred to [33]. The results are very similar.

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