Averaging (FEM) for Everything?

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The striking simplicity of averaging techniques and their amazing accuracy in too many numerical examples made them an extremly popular tool in scientific computing whenever finite elements might be useful. Given a dicrete flux p_h and an easily post-processed approximation $\mathcal{A}p_h$ to compute the error estimator $\eta_{\mathcal{A}} := \|p_h - \mathcal{A}p_h\|$. One does not even need an equation to emply that technique occasionally named after Zienkiewicz & Zhu.

The beginning of a mathematical justification of the error estimator $\eta_{\mathcal{A}}$ as a computable approximation of the (unknown) error $\|p - p_h\|$ involved the concept of super-convergence points. For highly structured meshes and a very smooth exact solution p, the error $\|p - \mathcal{A}p_h\|$ of the post-processed approximation $\mathcal{A}p_h$ may be (much) smaller than $\|p - p_h\|$ of the given p_h. Under the assumption that $\|p - \mathcal{A}p_h\| =$ h.o.t. is relatively sufficiently small, the triangle inequality immediately verifies reliability, i.e.,

$$\|p - p_h\| \leq C_{rel}\, \eta_{\mathcal{A}} + \text{ h.o.t.},$$

and efficiency, i.e.,

$$\eta_{\mathcal{A}} \leq C_{eff}\, \|p - p_h\| + \text{ h.o.t.},$$

of the averaging error estimator $\eta_{\mathcal{A}}$. However, the underlying assumptions essentially contradict the notion of adaptive grid refining for optimal experimental convergence rates when p is singular. Moreover, the proper treatment of boundary conditions lacks a serious inside.

The presentation reports on old and new arguments for reliability and efficiency in the above sense with multiplicative constants C_{rel} and C_{eff} and higher order terms h.o.t. Hilighted are the general class of meshes, averaging techniques, or finite element methods (conforming, nonconforming, and mixed elements) for elliptic PDEs. Numerical examples illustrate the amazing accuracy of $\eta_{\mathcal{A}}$. The presentation closes with a discussion on current developments and the limitations as well as the perspectives of averaging techniques.