Discrete Laplace-Beltrami Operator on Sphere and Optimal Spherical Triangulations

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Abstract

In this paper we first modify a widely used discrete Laplace Beltrami operator proposed by Meyer et al over triangular surfaces, and then establish some convergence results for the modified discrete Laplace Beltrami operator over the triangulated spheres. A sequence of spherical triangulations which is optimal in certain sense and leads to smaller truncation error of the discrete Laplace Beltrami operator and a sequence of hierarchical spherical triangulations are constructed. Truncation error bounds of the discrete Laplace Beltrami operator over the constructed triangulations are provided.

Key words: Laplace-Beltrami operator; Convergence; Spherical triangulation, Truncation error.

1 Introduction

In the computation of numerical weather forecast, partial differential equations (PDE) are often solved on the spherical triangular meshes. Traditional partitions of sphere using latitude and longitude lines are not desirable since the resulting meshes are not uniform, especially at the two poles. Therefore, the use of uniform triangulations in some sense is gaining popularity in the climate modeling community (see [5] [10]). On the other hand, many PDEs to be solved over sphere involve Laplace Beltrami operator (abbreviated as LB operator in this paper). A discrete version of the LB operator over spherical triangulation is therefore required. The aim of the this paper is to study the convergence of a widely used discrete LB operator proposed by Meyer et al [9] with our modification, and then construct a sequence of spherical triangulations which are uniform approximately and lead to smaller truncation error of the discrete LB operator. In several application areas, such as adaptive technique and multi-grid technique, hierarchical meshes are required. We therefore construct a sequence of triangulations which are hierarchical.

The convergence of the discrete LB operators is the foundation for the convergence analysis of some numerical simulation process of PDE which involves the LB operator. In paper [14], the author has reviewed several already used discrete LB operators over triangulated surface and study numerically as well as theoretically their convergent behavior. Attention has been focused on a family of discrete LB operators over triangulated surfaces, including Taubin's discretization (see [11], 1995; [12], 2000), Fujiwara's discretization (see

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[6], 1995), Desbrun et al's discretization (see [2], 1999), Mayer's discretization (see [8], 2001), Meyer et al's discretization (see [9],2002), and Desbrun et al's discretization (see [3], 2000). All these discretizations can be written in the following form:

$$\Delta_M f(p_i) = \sum_{j\in N(i)} w_{ij}(f(p_j) - f(p_i)), \tag{1.1}$$

where p_i and p_j are the vertices of the surface triangulation M, $N(i)$ is the index set of one-ring neighbors of vertex p_i, w_{ij} are some constants which depend only on geometric property of the mesh. In another development [15], the author has proposed several other discretizations that have convergence properties under various conditions. The convergence problems considered in [14] and [15] are for the triangulation of general surfaces. In this paper, we focus our attention on spherical triangulation. We expect that the particularity of the sphere would yield special and different convergence results. For some reasons that will be clear soon, we only consider in this paper the discretization proposed by Meyer et al (see [9],2002) with our modification.

The rest of the paper is organized as follows. In section 2, we describe briefly the LB operator and its discretizations , and then, in section 3 and 4, we give several theoretical results of the convergence for the discrete mean curvature and discrete LB operators over sphere, respectively. In section 5, we define the optimal spherical triangulation and describe a computational algorithm. Truncation errors of the discrete LB operators are discussed in section 6. Section 7 concludes the paper.

2 LB operator and its Discretization

Let $\mathcal{M} \subset I\!R^3$ be a two-dimensional manifold, and $\{U_\alpha, x_\alpha\}$ be the differentiable structure. The mapping x_α with $x \in x_\alpha(U_\alpha)$ is called a parameterization of $\mathcal{M}$ at x. Denoting the coordinate U_α as (ξ_1, ξ_2), then for $f \in C^2(\mathcal{M})$, the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$ applying to f is given by (see [1])

$$\begin{aligned}\Delta_{\mathcal{M}} f &= \frac{1}{\sqrt{\det(G)}} \sum_{ij} \frac{\partial}{\partial \xi_i}\left(g^{ij}\sqrt{\det(G)}\frac{\partial f}{\partial \xi_j}\right) \\ &= \frac{1}{\sqrt{\det(G)}}\left[\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}\right]\sqrt{\det(G)}G^{-1}\left[\begin{array}{cc} \frac{\partial f}{\partial \xi_1}, & \frac{\partial f}{\partial \xi_2}\end{array}\right]^T,\end{aligned} \tag{2.1}$$

where g^{ij} is defined by $G^{-1} = (g^{ij})_{ij}$,

$$G^{-1} = \frac{1}{\det(G)}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{bmatrix}, \quad G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{bmatrix}, \quad g_{ij} = \langle t_i, t_j\rangle$$

and $t_i = \frac{\partial x}{\partial \xi_i}$ are the tangent vectors. Let p be a surface point of $\mathcal{M}$. Then it is known that (see [13], page 151)

$$\Delta_{\mathcal{M}} p = -2H(p) \in I\!R^3, \tag{2.2}$$

where $H(p)$ is the mean curvature normal at p. i.e., $\|H(p)\|$ is the mean curvature, $H(p)/\|H(p)\|$ is the unit surface normal.

Let M be a triangulation of the surface $\mathcal{M}$ and $\{p_i\}_{i=1}^N$ be its vertex set. For a vertex p_i with valence n, denote by $N(i) = \{i_1, i_2, \cdots, i_n\}$ the set of the vertex indices of one-ring neighbors of p_i. We assume in the following that these $i_1, \cdots, i_n$ are arranged

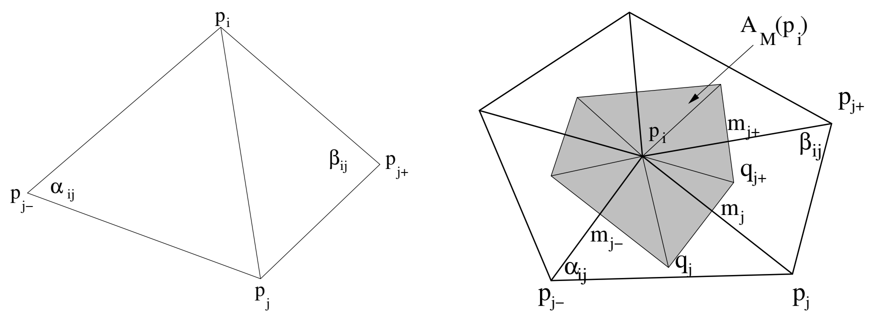

Fig 2.1: Left: The definition of the angles α_{ij} and β_{ij}. Right: The definition of the area $A_M(p_i)$.

such that the triangles $[p_i p_{i_k} p_{i_{k-1}}]$ and $[p_i p_{i_k} p_{i_{k+1}}]$ are in M, and $p_{i_{k-1}}$, $p_{i_{k+1}}$ opposite to the edge $[p_i p_{i_k}]$. Furthermore, we assume p_{i_1}, p_{i_2}, $\cdots$, p_{i_n} are in the counter clockwise order. For $j = i_k \in N(i)$, we use j_+ and j_- to denote i_{k+1} and i_{k-1}, respectively, for simplifying the notation. Furthermore, we use the following convention throughout the paper: $i_{n+1} = i_1$, $i_0 = i_n$.

A. Meyer et al's discretization. First let us introduce the discretization of $\Delta_{\mathcal{M}} f$ proposed by Meyer et al (see [9], 2002):

$$\Delta_M f(p_i) = \frac{1}{A_M(p_i)} \sum_{j \in N(i)} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} [f(p_j) - f(p_i)], \tag{2.3}$$

where α_{ij} and β_{ij} are the triangle angles as shown in Fig 2.1,

$$A_M(p_i) = \sum_{j \in N(i)} A[p_i q_j m_j] + A[p_i q_{j+} m_j] \tag{2.4}$$

is the sum of triangle areas for vertex p_i as shown in Fig 2.1 (right), where $m_j = \frac{1}{2}(p_i + p_j)$, q_j is the circumcenter of the triangle $[p_{j-} p_j p_i]$ if the triangle is non-obtuse. If the triangle is obtuse, q_j is chosen to be the midpoint of the edge opposite to the obtuse angle.

B. A different derivation. Now we give another derivation of the discrete LBO (2.3). First let us introduce Gauss theorem (see [7], page 330):

Let $\mathcal{M}$ be a Riemannian manifold. Let $\partial\mathcal{M}$ be the piecewise smooth boundary of $\mathcal{M}$. Let n be the unit outward normal vector field to the boundary, and X be a C^1 vector field on $\mathcal{M}$ with compact support. Then

$$\int_{\mathcal{M}} (\mathrm{div}_{\mathcal{M}} X) dv_{\mathcal{M}} = \int_{\partial\mathcal{M}} \langle X, n \rangle dv_{\partial\mathcal{M}}, \tag{2.5}$$

where $dv_{\mathcal{M}}$ and $dv_{\partial\mathcal{M}}$ denote the canonical metric on $\mathcal{M}$ and $\partial\mathcal{M}$, respectively.

Let f be a C^2 smooth function on $\mathcal{M}$, then $\nabla_{\mathcal{M}} f$ is a C^1 vector field on $\mathcal{M}$. Take $X = \nabla_{\mathcal{M}} f := [\ t_1,\ t_2\ ] G^{-1} [\frac{\partial f}{\partial \xi_1},\ \frac{\partial f}{\partial \xi_2}]^T$ in (2.5), then, since $\Delta_{\mathcal{M}} f = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} f$, we have

$$\int_{\mathcal{M}} \Delta_{\mathcal{M}} f dv_{\mathcal{M}} = \int_{\partial\mathcal{M}} \langle \nabla_{\mathcal{M}} f, n \rangle dv_{\partial\mathcal{M}}. \tag{2.6}$$

Discretizing (2.6) on the domain as shown in Fig. 2.1 (shaded part of the right figure), we obtain the following discretization of $\Delta_{\mathcal{M}}$:

$$\Delta_M f(p_i) = \frac{1}{A_M(p_i)} \sum_{j \in N(i)} c_{ij} \frac{f(p_j) - f(p_i)}{\|p_j - p_i\|}, \tag{2.7}$$

where $c_{ij} = a_{ij}\|q_j - m_j\| + b_{ij}\|q_{j+} - m_j\|$, $a_{ij} = \text{sgn}(\cot\alpha_{ij})$, $b_{ij} = \text{sgn}(\cot\beta_{ij})$. Note that this discretization is similar to the one proposed by Ringler and Randall [10]. The difference is that we evaluate the distance and area on triangles, they evaluate over the sphere. From the elementary geometry, we know that $\alpha_{ij} = \angle p_i q_j m_j$, $\beta_{ij} = \angle p_i q_{j+} m_j$. Hence,

$$\cot\alpha_{ij} = 2a_{ij}\frac{\|q_j - m_j\|}{\|p_j - p_i\|}, \ \cot\beta_{ij} = 2b_{ij}\frac{\|q_{j+} - m_j\|}{\|p_j - p_i\|}. \tag{2.8}$$

Therefore, (2.3) and (2.7) are exactly the same.

C. Modification. In the modified version of (2.3) or (2.7), the definition of $A_M(p_i)$ is different. First q_j is always chosen to be the circumcenter of the triangle $[p_{j-}p_jp_i]$ no matter the triangle is obtuse or not. The area $A_M(p_i)$ is then computed by

$$\begin{aligned} A_M(p_i) &= \sum_{j\in N(i)} a_{ij}A[p_iq_jm_j] + b_{ij}A[p_iq_{j+}m_j] \\ &= \sum_{j\in N(i)} \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{8}\|p_j - p_i\|^2, \end{aligned} \tag{2.9}$$

where the equalities

$$a_{ij}A[p_iq_jm_j] = \frac{\cot\alpha_{ij}\|p_j - p_i\|^2}{8}, \quad b_{ij}A[p_iq_{j+}m_j] = \frac{\cot\beta_{ij}\|p_j - p_i\|^2}{8} \tag{2.10}$$

are used (formula (2.9) has been obtained by Meyer et al in [9]). Therefore, our discretization of $\Delta_{\mathcal{M}} f$ is as follows:

$$\begin{aligned} \Delta_M f(p_i) &= \frac{1}{A_M(p_i)}\sum_{j\in N(i)} \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2}[f(p_j) - f(p_i)] \\ &= \frac{4\sum_{j\in N(i)}(\cot\alpha_{ij} + \cot\beta_{ij})[f(p_j) - f(p_i)]}{\sum_{j\in N(i)}(\cot\alpha_{ij} + \cot\beta_{ij})\|p_j - p_i\|^2}. \end{aligned} \tag{2.11}$$

Taking $f = p \in \mathcal{M}$ in (2.3), we have by (2.2) a discretization of mean curvature H at p_i

$$H_M(p_i) = \frac{1}{A_M(p_i)}\sum_{j\in N(i)} \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{4}(p_i - p_j). \tag{2.12}$$

Note that negative area is used if the angle α_{ij} or β_{ij} is great than $\pi/2$. Considering the fact that the triangle area $A[p_iq_jm_j]$ (or $A[p_iq_{j+}m_j]$) in (2.4) is taken to be zero if α_{ij} (or β_{ij}) is great than $\pi/2$, we can see from (2.10) that the signed triangle area $a_{ij}A[p_iq_jm_j]$ (or $b_{ij}A[p_iq_{j+}m_j]$) relates to the triangle angle α_{ij} (or β_{ij}) in a more reasonable and consistent way. The area expression (2.9) does not involve the computation of the circurmcenters. This makes the formulas (2.11) and (2.12) more efficient in various applications. More importantly, this modification makes $\Delta_M p_i$ converge to $\Delta_S p_i$ even if there are obtuse triangles around p_i (see next section). However, if the triangle $[p_ip_jp_{j-}]$ are $[p_ip_jp_{j+}]$ are too long and narrow (see Fig. 3.1), the negative triangle areas may be large enough so that the total area $A_M(p_i)$ is close to zero or even negative. If such a case happens, we suggest to flip the edge $[p_ip_j]$ (i.e., remove the edge $[p_ip_j]$ and then form a new edge $[p_{j-}p_{j+}]$). Note that the modified $\Delta_M f(p_i)$ and the original $\Delta_M f(p_i)$ are the same if all the triangles around p_i are acute. In the following, $\Delta_M f(p_i)$ stands for the modified one if no specific interpretation.

3 Convergence of Discrete Mean Curvatures

First we present an interesting result on the discrete mean curvature approximation $H_M(p_i)$ which says that this approximation is exact for regularly distributed vertices.

Theorem 3.1 *Let p_i be a vertex of a triangulation M of the sphere $S = \{p \in \mathbb{R}^3 : \|p\| = R\}$ and p_j, $j \in N(i)$, be regularly distributed vertices around p_i. Then*

$$\Delta_M p_i := \frac{1}{A_M(p_i)} \sum_{j \in N(i)} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} (p_j - p_i) = \Delta_S p_i. \tag{3.1}$$

Proof. Without loss of generality, we may assume that the sphere radius R is one, and $p_i = (0, 0, 1)^T$. Then the sphere can be expressed parametrically by

$$\begin{cases} x &= x \\ y &= y \\ z &= \sqrt{1 - x^2 - y^2} \end{cases} \tag{3.2}$$

around p_i, we further assume that $p_j = \left(h \cos \theta_j, h \sin \theta_j, \sqrt{1 - h^2}\right)^T$, $j \in N(i)$ with $\theta_j - \theta_{j-} = \frac{2\pi}{n}$ and $h = \frac{1}{2} r \sqrt{4 - r^2}$, $r = \|p_j - p_i\|$. Then

$$\alpha_{ij} = \alpha_{ik} = \beta_{ij} = \beta_{ik}, \quad \forall j, k \in N(i).$$

Since

$$\begin{aligned} & p_j - p_i = (h \cos \theta_j, h \sin \theta_j, \sqrt{1 - h^2} - 1)^T, \\ & \|p_j - p_i\|^2 = 2(1 - \sqrt{1 - h^2}), \\ & \sum_{j \in N(i)} \cos \theta_j = 0, \quad \sum_{j \in N(i)} \sin \theta_j = 0, \end{aligned} \tag{3.3}$$

we have

$$\Delta_M(p_i) = -2[0, 0, 1]^T = -2H = \Delta_S(p_i).$$

◇

Remark. If the vertices $p_j, j \in N(i)$, around p_i are not regularly distributed, the equality $\Delta_M p_i = \Delta_S p_i$ does not hold in general, but $\Delta_M p_i$ is an approximation to $\Delta_S p_i$. Specifically, we have the following result.

Theorem 3.2 *Let M be a triangulation of the sphere $S = \{p \in \mathbb{R}^3 : \|p\| = R\}$. Let p_i be a vertex of the triangulation and p_j, $j \in N(i)$, be the neighbor vertices around p_i, Then*

$$\Delta_M p_i = \Delta_S p_i + O(r), \quad \text{as} \quad r \to 0, \tag{3.4}$$

where $r = \max_{j \in N(i)} \|p_j - p_i\|$.

Proof. Again, we assume the radius R of the sphere is one and $p_i = (0, 0, 1)^T$. Then there exist $\bar{p}_j = h_j (\cos \theta_j, \sin \theta_j)^T$, $j \in N(i)$, such that $p_j = \left(h_j \cos \theta_j, h_j \sin \theta_j, \sqrt{1 - h_j^2}\right)^T$, where

$$h_j = \frac{1}{2} r_j \sqrt{4 - r_j^2}, \quad r_j = \|p_j - p_i\|.$$

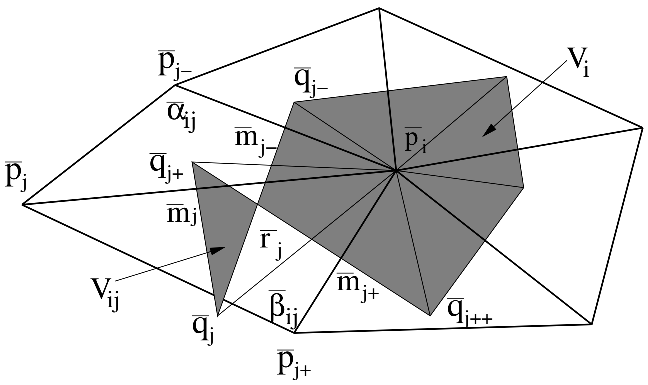

Fig 3.1: Nonempty region V_{ij} is formed if the line segments $[\bar{q}_{j-}\bar{q}_j]$ and $[\bar{q}_{j+}\bar{q}_{j++}]$ intersect.

Using Taylor expansion, we have

$$p_j = \left(h_j \cos\theta_j, h_j \sin\theta_j, 1 - \frac{1}{2}h_j^2 + O(h^4) \right)^T, \quad j \in N(i),$$

where $h = \max_{j\in N(i)} h_j$. Hence

$$p_j - p_i = \left(h_j \cos\theta_j, h_j \sin\theta_j, -\frac{1}{2}h_j^2 + O(h^4) \right)^T, \ \|p_j - p_i\|^2 = h_j^2 + O(h^4).$$

Now we compute $\cot\alpha_{ij}$. Let $u = p_{j-} - p_i, \quad v = p_{j-} - p_j$. Then

$$\begin{aligned} \|u\|^2 &= h_{j-}^2 + O(h_{j-}^4), \\ \|v\|^2 &= h_{j-}^2 + h_j^2 - 2h_{j-}h_j \cos\delta_j + O(h^4), \\ (u, v) &= h_{j-}^2 - h_{j-}h_j \cos\delta_j + O(h^4), \end{aligned}$$

where $\delta_j = \theta_j - \theta_{j-}$ $(0 < \delta_j < \pi)$. Hence

$$\cot\alpha_{ij} = \frac{(u, v)}{\sqrt{\|u\|^2\|v\|^2 - (u, v)^2}} = \frac{h_{j-} - h_j \cos\delta_j}{h_j \sin\delta_j} + O(h^2). \tag{3.5}$$

Similarly,

$$\cot\beta_{ij} = \frac{h_{j+} - h_j \cos\delta_{j+}}{h_j \sin\delta_{j+}} + O(h^2).$$

Therefore,

$$\Delta_M p_i = \frac{4\sum_{j\in N(i)} \left[\frac{h_{j-} - h_j \cos\delta_j}{\sin\delta_j} + \frac{h_{j+} - h_j \cos\delta_{j+}}{\sin\delta_{j+}} \right] \left[\ \cos\theta_j, \sin\theta_j, -\frac{1}{2}h_j \ \right]^T}{\sum_{j\in N(i)} \left[\frac{h_{j-} - h_j \cos\delta_j}{\sin\delta_j} + \frac{h_{j+} - h_j \cos\delta_{j+}}{\sin\delta_{j+}} \right] h_j} + O(h). \tag{3.6}$$

It is easy to see that the third component of $\Delta_M p_i$ is -2 + O(h). To complete the proof of the theorem, we now illustrate that the first and second components of the numerator in (3.6) are zeros.

Let $\bar{p}_i$ and $\bar{p}_j$, $j \in N(i)$, be the projections of p_i and p_j, respectively, on the xy-plane by taking their third components to be zeros. Let V_i and V_{ij} be the regions formed by consecutive connecting the points $\bar{q}_{j-}$, $\bar{q}_j$, $\bar{q}_{j+}, \cdots$ (see Fig. 3.1), where $\bar{q}_j$ is the circumcenter of the triangle $[\bar{p}_{j-}\bar{p}_j\bar{p}_i]$. $V_{ij} = [\bar{q}_j\bar{q}_{j+}\bar{r}_j]$ is nonempty if the two line segments

$[\bar{q}_{j-}\bar{q}_j]$ and $[\bar{q}_{j+}\bar{q}_{j++}]$ intersect at $\bar{r}_j$, and empty otherwise. Consider the formula (2.6) over the planar domains V_i and V_{ij}:

$$\int_{V_i} \Delta f dxdy - \sum_{j\in N(i),\mathring{V}_{ij}\neq\emptyset} \int_{V_{ij}} \Delta f dxdy = \int_{\partial V_i} \nabla f^T n ds - \sum_{j\in N(i),\mathring{V}_{ij}\neq\emptyset} \int_{\partial V_{ij}} \nabla f^T n ds, \tag{3.7}$$

where $\mathring{V}_{ij}$ denotes the interior of V_{ij}. Since the domains V_i and V_{ij} are flat, the Laplace-Beltrami operator is Laplacian. Take $f = (x, y)^T$, a linear vector function, then $\Delta f = 0$, ∇f is a constant vector. Hence the left-handed side of (3.7) is a zero vector. Since normal n is a constant vector on each line segment of the domain boundaries, $(\nabla f)^T n$ is a constant vector. Therefore, the right-handed side integrals of (3.7) can be computed exactly. Let $\bar{m}_j = \frac{\bar{p}_i+\bar{p}_j}{2}$, and the angles $\bar{\alpha}_{ij}$ and $\bar{\beta}_{ij}$ be defined as in Fig. 3.1. Then it is easy to see that the right-handed side of (3.7) around the edge $[\bar{p}_i\bar{p}_j]$ is

$$\begin{aligned}
&\int_{[\bar{q}_{j-}\bar{r}_j]} \nabla f^T n ds + \int_{[\bar{q}_{j++}\bar{r}_j]} \nabla f^T n ds - \int_{\partial V_{ij}} \nabla f^T n ds \\
&= [\|\bar{q}_{j-} - \bar{m}_{j-}\| + \|\bar{m}_{j-} - \bar{r}_j\|] \frac{\bar{p}_{j-} - \bar{p}_i}{\|\bar{p}_{j-} - \bar{p}_i\|} \\
&+ [\|\bar{q}_{j++} - \bar{m}_{j+}\| + \|\bar{m}_{j+} - \bar{r}_j\|] \frac{\bar{p}_{j+} - \bar{p}_i}{\|\bar{p}_{j+} - \bar{p}_i\|} \\
&- \|\bar{q}_{j+} - \bar{r}_j\| \frac{\bar{p}_i - \bar{p}_{j+}}{\|\bar{p}_i - \bar{p}_{j+}\|} - \|\bar{q}_j - \bar{r}_j\| \frac{\bar{p}_i - \bar{p}_{j-}}{\|\bar{p}_i - \bar{p}_{j-}\|} - \|\bar{q}_{j+} - \bar{q}_j\| \frac{\bar{p}_j - \bar{p}_i}{\|\bar{p}_j - \bar{p}_i\|} \\
&= [\|\bar{q}_{j-} - \bar{m}_{j-}\| + \|\bar{m}_{j-} - \bar{q}_j\|] \frac{\bar{p}_{j-} - \bar{p}_i}{\|\bar{p}_{j-} - \bar{p}_i\|} \\
&+ [\|\bar{q}_{j++} - \bar{m}_{j+}\| + \|\bar{m}_{j+} - \bar{q}_{j+}\|] \frac{\bar{p}_{j+} - \bar{p}_i}{\|\bar{p}_{j+} - \bar{p}_i\|} \\
&+ \left[\mathrm{sgn}(\cot\bar{\alpha}_{ij})\|\bar{q}_j - \bar{m}_j\| + \mathrm{sgn}(\cot\bar{\beta}_{ij})\|\bar{m}_j - \bar{q}_{j+}\|\right] \frac{\bar{p}_j - \bar{p}_i}{\|\bar{p}_j - \bar{p}_i\|}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &= \sum_{j\in N(i)} \left[\mathrm{sgn}(\cot\bar{\alpha}_{ij})\|\bar{q}_j - \bar{m}_j\| + \mathrm{sgn}(\cot\bar{\beta}_{ij})\|\bar{q}_{j+} - \bar{m}_j\|\right] \frac{\bar{p}_j - \bar{p}_i}{\|\bar{p}_j - \bar{p}_i\|} \\
&= \sum_{j\in N(i)} \frac{h_j(\cot\bar{\alpha}_{ij} + \cot\bar{\beta}_{ij})}{2} \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix}.
\end{aligned} \tag{3.8}$$

Parallel to the derivation of (3.5), now we can derive that

$$\cot\bar{\alpha}_{ij} = \frac{h_{j-} - h_j\cos\delta_j}{h_j\sin\delta_j}, \quad \cot\bar{\beta}_{ij} = \frac{h_{j+} - h_j\cos\delta_{j+}}{h_j\sin\delta_{j+}}.$$

Substitute these into (3.8), we obtain that the first and second components of the numerator in (3.6) are zeros. Therefore, the theorem is proved by noting that $O(h) = O(r)$ as $r \to 0$. $\diamond$

Remark. The theorem says that $\Delta_M p_i$ converges to $\Delta_S p_i$ with a linear rate. However, the numerical experiment shows that the convergence rate is $O(r^2)$ instead of $O(r)$. Hence there still exists a room for improving the theorem on the convergence rate.

4 Convergence of Discrete Laplace Beltrami Operator

Though $\Delta_M p_i = \Delta_S p_i$ exactly under the condition of theorem 3.1, we cannot have the similar result $\Delta_M f(p_i) = \Delta_S f(p_i)$, because the discrete function data $f(p_j)$ cannot determine the behavious of f around p_i. However, convergence result can be established based on the following lemma.

Lemma 4.1 *Let f be a sufficiently smooth function on $S = \{p \in I\!R^3 : \|p\| = R\}$. Assume f is extended smoothly to a 3D neighborhood of S. Then*

$$\Delta_S f(p) = (\Delta_S p)^T \nabla f(p) + \Delta f(p) - p^T \mathrm{H} f(p) p, \quad p \in S, \tag{4.1}$$

where ∇, Δ and H are classical gradient, Laplace and Hessian operators, respectively.

Proof. Recall that

$$\Delta_S f(p) = \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}\right] \sqrt{g} G^{-1} \nabla (f \circ p),$$

where

$$\nabla (f \circ p) = \left[\frac{\partial f(p(\xi_1, \xi_2))}{\partial \xi_1}, \frac{\partial f(p(\xi_1, \xi_2))}{\partial \xi_2}\right] = [t_1, t_2]^T \nabla f(p),$$

and $p(\xi_1, \xi_2)$ is a parameterization of S. Let

$$R_i = e_i^T G^{-1} [t_1, t_2]^T, \quad \text{with} \quad e_1 = [1, 0]^T, \; e_2 = [0, 1]^T.$$

Then

$$\begin{aligned} \Delta_S f(p) &= \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}\right] \left(\sqrt{g} G^{-1} [t_1, t_2]^T\right) \nabla f(p) + R_1 \frac{\partial}{\partial \xi_1} \nabla f(p) + R_2 \frac{\partial}{\partial \xi_2} \nabla f(p) \\ &= (\Delta_S p)^T \nabla f(p) + R_1 H f(p) t_1 + R_2 H f(p) t_2. \end{aligned} \tag{4.2}$$

Without loss of generality, we may assume S is represented as (3.2). Then $t_1 = [1, 0, -x/\sqrt{1 - x^2 - y^2}]^T$, $t_2 = [0, 1, -y/\sqrt{1 - x^2 - y^2}]^T$, and

$$G^{-1} = \begin{bmatrix} 1 - x^2 & -xy \\ -xy & 1 - y^2 \end{bmatrix},$$

$$R_1 = [y^2 + z^2, -xy, -xz], \quad R_2 = [-xy, x^2 + z^2, -yz].$$

Substitute these into (4.2), we obtain (4.1). $\diamond$

Theorem 4.1 *Under the condition of Theorem 3.1, let f be a sufficiently smooth function on S. Then*

$$\Delta_M f(p_i) = \Delta_S f(p_i) + O(r) \quad \text{as} \quad r \to 0. \tag{4.3}$$

Proof. It follows from the proof of Theorem 2.1 that

$$\frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2 A_M(p_i)} = \frac{2}{n(1 - \sqrt{1 - h^2})} = \frac{1}{O(r^2)}, \tag{4.4}$$

where $n = |N(i)|$. Now we need to compute $f(p_j) - f(p_i)$. To do so, we assume f is extended smoothly to the neighborhood of surface S. Such an extension is always possible. For instance, a simple way to do so is to define f as a constant function in the normal direction of the sphere. Using Taylor expansion of $f(p_i)$ at p_i, we have

$$f(p_j) - f(p_i) = (p_j - p_i)^T \nabla f(p_i) + \frac{1}{2}(p_j - p_i)^T H f(p_i)(p_j - p_i) + O(r_j^3), \tag{4.5}$$

where $r_j = \|p_j - p_i\|$. Using (4.4), we have

$$\begin{aligned} \Delta_M f(p_i) &= \frac{2}{n(1-\sqrt{1-h^2})}\Big[\sum_{j\in N(i)} (p_j - p_i)\Big]^T \nabla f(p_i) \\ &+ \frac{1}{n(1-\sqrt{1-h^2})} \sum_{j\in N(i)} (p_j - p_i)^T \nabla H(p_i)(p_j - p_i) + O(r), \end{aligned} \tag{4.6}$$

where $r = \max_{j\in N(i)} r_j$. To simplify the notation, let $(x_j, y_j, z_j) = (h\cos\theta_j, h\sin\theta_j, \sqrt{1-h^2} - 1)$, then we have

$$\sum_{j\in N(i)} (x_j^2,\ y_j^2,\ z_j^2) = n\left[\frac{h^2}{2},\ \frac{h^2}{2},\ \left(\sqrt{1-h^2}-1\right)^2\right], \tag{4.7}$$

$$\sum_{j\in N(i)} (x_j y_j,\ x_j z_j,\ y_j z_j) = (0,\ 0,\ 0). \tag{4.8}$$

Hence

$$\sum_{j\in N(i)} (p_j - p_i)^T \nabla H(p_i)(p_j - p_i) = \frac{nh^2}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + n\left(\sqrt{1-h^2}-1\right)^2 \frac{\partial^2 f}{\partial z^2} + O(h^3).$$

Substitute this into (4.6) and notice that

$$1 - \sqrt{1-h^2} = \frac{h^2}{2} + O(h^4), \quad \left(\sqrt{1-h^2}-1\right)^2 = O(h^4),$$

we have

$$\Delta_M f(p_i) = (\Delta_S p_i)^T \nabla f(p_i) + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + O(h). \tag{4.9}$$

To complete the proof of the theorem, we need to show that

$$\Delta_S f(p_i) = (\Delta_S p_i)^T \nabla f(p_i) + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}. \tag{4.10}$$

This is true by taking $p = p_i = (0,0,1)^T$ in Theorem 4.1. Therefore, the theorem is proved. $\diamond$

Remark. Theorem 4.1 says that $\Delta_M f(p_i)$ converges to $\Delta_S f(p_i)$ as $r \to 0$ if the vertices p_j around p_i are regularly distributed. Considering the convergence result $\Delta_M p_i \to \Delta_M p_i$ in Theorem 3.2 that holds without any regularity condition, one may ask if $\Delta_M f(p_i)$ converges to $\Delta_S f(p_i)$ in general. The answer is no. To illustrate this, we present the following example.

Example 4.1. Take $p_i = (0,0,1)^T$ on the unit sphere $S = \{p \in I\!R^3 : \|p\| = 1\}$,

$$\begin{aligned} p_j &= \left(r_{jm}\cos\theta_j, r_{jm}\sin\theta_j, \sqrt{1-r_{jm}}\right)^T \in S, \\ r_{jm} &= 2^{-m}(1 + 0.2 * j\%3), \quad \theta_j = \frac{\pi}{4}(1 + 0.2 * j\%4), \end{aligned}$$

Table 4.1: The errors $E_L = |\Delta_M f(p_i) - \Delta_S f(p_i)|$ and $E_H = \|\Delta_M p_i - \Delta_M p_i\|$

m	3	4	5	6	7	8	9	10
E_H	5.0664e-02	5.0116e-02	4.9981e-02	4.9948e-02	4.9939e-02	4.9937e-02	4.9937e-02	4.9937e-02
E_H	6.2684e-05	1.5427e-05	3.8418e-06	9.5952e-07	2.3982e-07	5.9952e-08	1.4988e-08	3.7469e-09
E_L	1.3538e-01	1.1384e-01	1.0948e-01	1.0891e-01	1.0902e-01	1.0918e-01	1.0928e-01	1.0934e-01
E_L	2.8599e-02	4.8720e-02	5.2699e-02	5.3168e-02	5.3022e-02	5.2854e-02	5.2746e-02	5.2686e-02

for $j = 1, 2, \cdots, 8$, where $j\%k$ stands for j modulo k. Taking $f(x, y, z) = \cos x + \sin y + \exp z$, and $m = 3, 4, 5, \cdots$, we compute the errors $E_L := |\Delta_M f(p_i) - \Delta_S f(p_i)|$ and $E_H := \|\Delta_M p_i - \Delta_S p_i\|$. These errors are listed in the Table 4.1. The first and second rows are the errors E_H for the original version and our modified version of $\Delta_M p_i$, respectively. Non-convergent and convergent properties are observed for the original version and modified version, respectively. Third and fourth rows are the errors E_L for the original version and the modified version of $\Delta_M f(p_i)$, respectively. Both of them do not converge, but the modified version has smaller errors. It is easy to see that, for the modified version, though $\Delta_M p_i$ converges to $\Delta_S p_i$ in the speed $(\frac{1}{2})^2$ approximately, the errors E_L in the fourth row nearly do not decrease as m increasing. However, we have the following result.

Theorem 4.2 *Suppose $\tilde{p}_j$ is a perturbation of p_j, which is defined by Theorem 4.1, satisfying*

$$\|\tilde{p}_j - p_j\| = o(r^2) \quad \text{as} \quad r \to 0. \tag{4.11}$$

Suppose f is sufficiently smooth function over S. Then

$$\lim_{r\to 0} \frac{1}{\tilde{A}_M(p_i)} \sum_{j\in N(i)} \frac{\cot\tilde{\alpha}_{ij} + \cot\tilde{\beta}_{ij}}{2}[f(\tilde{p}_j) - f(p_i)] = \Delta_S f(p_i),$$

where $\tilde{\alpha}_{ij}, \tilde{\beta}_{ij}$ and $\tilde{A}_M(p_i)$ are defined as α_{ij}, β_{ij} and $A_M(p_i)$ but from the perturbed data.

Proof. From the proof of Theorem 4.1, we can see that

$$\frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2A_M(p_i)} = \frac{1}{O(h^2)}.$$

Hence, if we perturb p_j such that (4.11) is satisfied. Then

$$\frac{\cot\tilde{\alpha}_{ij} + \cot\tilde{\beta}_{ij}}{2\tilde{A}_M(p_i)} = \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2A_M(p_i)} + \varepsilon_{ij},$$

$$f(\tilde{p}_j) - f(p_i) = f(p_j) - f(p_i) + \delta_{ij},$$

where $\varepsilon_{ij} = \frac{o(h)}{O(h^2)}$, $\delta_{ij} = o(h^2)$. Hence

$$\begin{aligned}
\sum_{j\in N(i)} \frac{\cot\tilde{\alpha}_{ij} + \cot\tilde{\beta}_{ij}}{2\tilde{A}_M(p_i)}[f(\tilde{p}_j) - f(p_i)] &= \sum_{j\in N(i)} \left[\frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2A_M(p_i)} + \varepsilon_{ij}\right][f(p_j) - f(p_i) + \delta_{ij}] \\
&= \sum_{j\in N(i)} \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2A_M(p_i)}[f(p_j) - f(p_i)] + \sum_{j\in N(i)} \varepsilon_{ij}[f(p_j) - f(p_i)] \\
&+ \sum_{j\in N(i)} \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2A_M(p_i)}\delta_{ij} + \sum_{j\in N(i)} \varepsilon_{ij}\delta_{ij} \\
&= \Delta_S f(p_i) + o(1),
\end{aligned}$$

where $\lim_{h\to 0} o(1) = 0$. ◇

5 Optimal Spherical Triangulations and their Computations

We now want to define a spherical triangulation that yields the least truncation error for the Laplace Beltrami operator. Since $\Delta_S p_i = -2H(p_i) = -2p_i$, we propose the following definition:

Definition 5.1 *Let M be a triangulation of the unit sphere S, $\{p_i\}_{i=1}^N$ be the vertex set. If for each vertex p_i, $i = 1, \cdots, N$, we have*

$$\frac{1}{A_M(p_i)} \sum_{j \in N(i)} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} (p_j - p_i) = -2p_i \tag{5.1}$$

Then we say that the triangulation M is optimal.

Regarding vertices p_i, $i = 1, \cdots, N$, as unknowns, (5.1) is a system of nonlinear equations. For any given topology of sphere triangulation, the optimal triangulation may not exist. However, for a special class of spherical triangulations. The optimal triangulations do exist. This class of triangulations are generated recursively as follows: Starting with an icosahedron inscribed in the unit sphere, subdivide recursively each triangle into four by dividing each edge into two at the middle point, and then project the middle point to the unit sphere in the normal direction. This process generates a sequence of triangulations that have $20, 80, 320, 1280, 5120, 20480, 81820, \cdots$, triangles, respectively. We denote these triangulations as Gen(0), Gen(1), $\cdots$. The corresponding optimal triangulations, denoted as Opt(0), Opt(1), $\cdots$, are computed using the following algorithm:

Algorithm 1. Denote the vertex set of Gen(k) as $\left\{p_i^{(0)}\right\}_{i=1}^{N_k}$, and set $l = 0$.

1. Compute new vertices by

$$p_i^{(l+1)} = \frac{1}{A_M(p_i^{(l)})} \sum_{j \in N(i)} \frac{\cot \alpha_{ij}^{(l)} + \cot \beta_{ij}^{(l)}}{4} (p_i^{(l)} - p_j^{(l)}), \quad \text{if} \quad v(p_i^{(l)}) = 6,$$

   for $i = 1, \cdots, N_k$, where $\cot \alpha_{ij}^{(l)}$, $\cot \beta_{ij}^{(l)}$ are computed using vertex data $\left\{p_i^{(l)}\right\}_{i=1}^{N_k}$, $v(p_i^{(l)})$ denotes the valence of the vertex $p_i^{(l)}$.

2. Project $p_i^{(l+1)}$ to the unit sphere in the normal direction, for $i = 1, \cdots, N_k$.

3. If $\max_{1 \le i \le N_k} \|p_i^{(l+1)} - p_i^{(l)}\|$ is less than a given ϵ (we choose $\epsilon = 10^{-20}$, using long double precision arithmetic operations), terminate and $\left\{p_i^{(l+1)}\right\}_{i=1}^{N_k}$ is the obtained vertex set; otherwise, increase l by one and return to step 1.

In a few recent publications, Du et al [4, 5] have introduced *Spherical Centroidal Voronoi Tessellation* (SCVT). A Voronoi tessellation $\{V_i\}_{i=1}^N$ of sphere S from the generators $\{p_i\}_{i=1}^N \subset S$ is called SCVT if and only if p_i is the constrained mass centroid of the Voronoi region V_i for $i = 1, \cdots N$. The constrained mass centroid p^c of the Voronoi region V_i is defined by the solution of the problem:

$$\min_{p \in V_i} F(p), \quad \text{where} \quad F(p) = \int_{V_i} \|y - p\|^2 ds(y).$$

The SCVT can be computed by begining with Gen(k), and moving the generating points to the constrained mass centroids of their corresponding Voronoi region. With the new set of generators, Voronoi regions are recomputed and the generators are moved again to the constrained mass centroid of the new Voronoi regions. This algorithm is called Lloyd's method (see [4, 5]). Let Dvt(k) denote the triangulations produced by the dual of SCVT generated from Gen(k) by the modified Lloyd's algorithm. The computations show that Dvt(k) and Opt(k) are very close to each other (see Table 5.1), though they are not exactly the same. The modified Lloyd's algorithm is the same as Lloyd's algorithm except that the vertices with valence 5 are fixed during the iteration. Fixing vertices with valence 5 in Algorithm 1 and Lloyd algorithm makes the solutions Opt(k) and Dvt(k) unique. Furthermore, this change to the Lloyd's algorithm leads to a faster convergence rate. In Table 5.1, dis(Opt(k),Dvt(k)) $:= \max \|p_i^{opt} - p_i^{dvt}\|$ are listed, where p_i^{opt} and p_i^{dvt} are the vertices of Opt(k) and Dvt(k). The distances between Opt(k) and Gen(k) are also presented in this table.

Note that Algorithm 1 for computing Opt(k) is similar to the Lloyd's algorithm for computing Dvt(k). However, this algorithm is much easier to implement, since no integrations are involved. Fig. 5.1 shows the first six triangular meshes Opt(k), $k = 0, 1, \cdots, 5$.

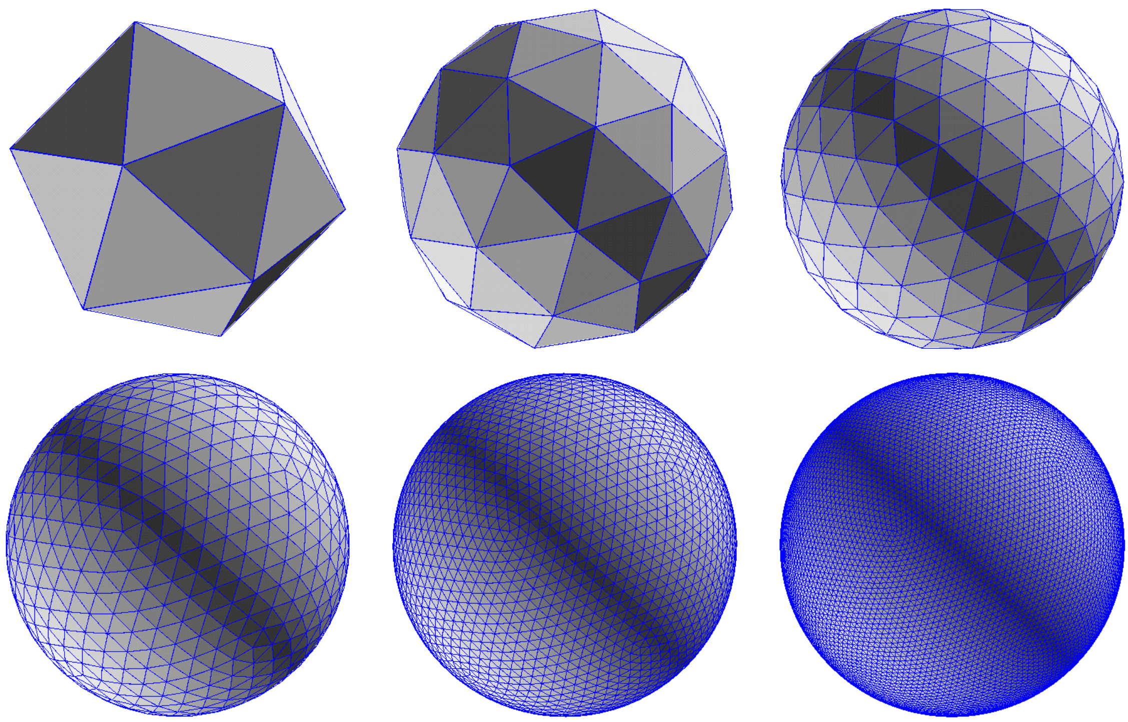

Fig 5.1: Spherical triangular meshes Opt(k) for $k = 0, 1, \cdots, 5$

Table 5.1: The distances between Opt(k) and Dvt(k), Hie(k) and Gen(k)

k	1	2	3	4	5	6	7
dis(Opt(k),Dvt(k))	0.0	3.1318e-06	6.7357e-07	1.0425e-07	2.4215e-08	6.1082e-09	2.0107e-09
dis(Opt(k),Hie(k))	0.0	1.5701e-16	2.0032e-03	3.2097e-03	3.5463e-03	3.6173e-03	3.6334e-03
dis(Opt(k),Gen(k))	0.0	1.5237e-02	1.5311e-02	1.5340e-02	1.5350e-02	1.5353e-02	1.5354e-02

Hierarchical Triangulation. Let $V(M)$ denote the vertex set of a triangular mesh M. Hierarchical triangulations of a sphere is a sequence triangulations $\{\text{Hie(k)}\}_{k=0}^{\infty}$, such that $V(\text{Hie(k)}) \subset V(\text{Hie(k+1)})$. Obviously, $\{\text{Gen(k)}\}_{k=0}^{\infty}$ is a hierarchical triangulation sequence. What we want to obtained is the triangulations which make equation (5.1) being satisfied approximately. Now we compute Hie(k) as follows: Refine the mesh Hie(k-1) in the same way as computing Gen(k) from Gen(k-1). Modify the refined mesh using Algorithm 1 for the newly added vertices (the vertices of Hie(k-1) are fixed). The distances between Opt(k) and Hie(k) are presented in Table 5.1.

6 Truncation Errors of Laplace Beltrami Operator

Let p_j, $j \in N(i)$, be the neighbor vertices of p_i. Then plug the Taylor expansion (4.5) of f into (1.1), we obtain an expansion of $\Delta_M f(p_i)$:

$$\Delta_M f(p_i) = \nabla f(p_i)^T \sum_{j\in N(i)} w_{ij}(p_j - p_i) + \frac{1}{2}\sum_{j\in N(i)} w_{ij}(p_j - p_i)^T Hf(p_j - p_i) + O(r), \quad (6.1)$$

where $r = \max_{j\in N(i)} r_j$, $r_j = \|p_j - p_i\|$,

$$w_{ij} = \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2A_M(p_i)} = \frac{c_{ij}}{A_M(p_i)\|p_j - p_i\|}. \quad (6.2)$$

For $p_i \in M$, let $\tilde{q}_j$ be n ($n = |N(i)|$) regularly distributed vertices, on the unit sphere, around $\tilde{q}_0 = (0,0,1)^T$ with $\|\tilde{q}_j - \tilde{q}_0\| = \tilde{r}$, $\tilde{r} = \frac{1}{n}\sum_{j\in N(i)} r_j$. Then from (4.4), (4.7) and (4.8) we have

$$\sum_{j\in N(i)} \tilde{w}_{ij}(\tilde{q}_j - \tilde{q}_0)(\tilde{q}_j - \tilde{q}_0)^T = \frac{1}{1-\sqrt{1-h^2}} \begin{bmatrix} h^2 & 0 & 0 \\ 0 & h^2 & 0 \\ 0 & 0 & 2(1-\sqrt{1-h^2})^2 \end{bmatrix}, \quad (6.3)$$

where $\tilde{w}_{ij}$ is computed from $\tilde{q}_0$, $\tilde{q}_j$ using (6.2), $h = \frac{1}{2}\tilde{r}\sqrt{4-\tilde{r}^2}$. Let $R(p_i) \in I\!R^{3\times 3}$ be a rotation matrix, such that $R(p_i)p_i = \tilde{q}_0$. Then $R(p_i)$ is an orthogonal matrix. Note that $\tilde{w}_{ij}$ is invariant under the transform $R(p_i)$. Let $\tilde{p}_j = R(p_i)^T \tilde{q}_j$. Then from (6.3), we have

$$\begin{aligned} \sum_{j\in N(i)} \tilde{w}_{ij}(\tilde{p}_j - p_i)(\tilde{p}_j - p_i)^T &= \frac{h^2}{1-\sqrt{1-h^2}} I + \frac{2(1-\sqrt{1-h^2})^2 - h^2}{1-\sqrt{1-h^2}} p_i p_i^T \\ &= \left(2 - \frac{1}{2}\tilde{r}^2\right) I + \left(\frac{3}{2}\tilde{r}^2 - 2\right) p_i p_i^T \qquad (6.4) \\ &= L(\tilde{r}, p_i) \end{aligned}$$

where $L(\tilde{r}, p_i) \in I\!R^{3\times 3}$ is the right-handed side matrix of (6.4). Using Theorem 3.1, Theorem 4.1 and the expansion (6.1), we have

$$\begin{aligned} |\Delta_M f(p_i) - \Delta_S f(p_i)| \;&\le\; \Big|\sum_{j\in N(i)} w_{ij}[f(p_j) - f(p_i)] - \sum_{j\in N(i)} \tilde{w}_{ij}[f(\tilde{p}_j) - f(p_i)]\Big| \\ &+ \Big|\sum_{j\in N(i)} \tilde{w}_{ij}[f(\tilde{p}_j) - f(p_i)] - \Delta_S f(p_i)\Big| \\ &\le \Big|\Big(\sum_{j\in N(i)} w_{ij}(p_j - p_i) + 2p_i\Big)^T \nabla f(p_i)\Big| + O(\tilde{r}) \\ &+ \frac{1}{2}\Big|\sum_{j\in N(i)} w_{ij}(p_j - p_i)^T Hf(p_i)(p_j - p_i) - \sum_{j\in N(i)} \tilde{w}_{ij}(\tilde{p}_j - p_i)^T Hf(p_i)(\tilde{p}_j - p_i)\Big| \\ &\le E_1(M, p_i)\|\nabla f(p_i)\|_l + \frac{1}{2}E_2(M, p_i)\|Hf(p_i)\|_l + O(\tilde{r}) \end{aligned}$$

where

$$E_1(M,p_i) = \Big\| \sum_{j\in N(i)} w_{ij}(p_j-p_i)^T + 2p_i) \Big\|_\infty,$$

$$E_2(M,p_i) = \Big\| \sum_{j\in N(i)} w_{ij}(p_j-p_i)(p_j-p_i)^T - L(\tilde{r},p_i) \Big\|_\infty,$$

and the l-normal for a matrix $[a_{ij}]$ is defined by $\sum_{ij}|a_{ij}|$. If the neighbor vertices of p_i are regularly distributed, then both $E_1(M,p_i)$ and $E_2(M,p_i)$ vanish. Hence these two quantities measure how far the neighbor vertices away from the regularly distributed vertex positions. Let $E_1(M) = \max_i E_1(M,p_i)$, $E_2(M) = \max_i E_2(M,p_i)$ for a given mesh M. Table 6.1 lists E_1 and E_2 for the triangulations mentioned in the last section. Table 6.2 lists the mean errors ME_1 and ME_2 for these triangulations. For a mesh M, the mean error ME_i(M) are defined as follows

Table 6.1: Errors E_1 and E_2 of Δ_M for Gen(k), Hie(k), Dvt(k) and Opt(k)

k	1	2	3	4	5	6	7
E_1(Gen(k))	4.09612e-16	3.2547e-02	1.6409e-02	8.2103e-02	4.1053e-03	2.0527e-03	1.0263e-03
E_1(Hie(k))	4.09612e-16	2.3852e-17	1.7880e-03	1.5472e-03	1.1011e-03	7.7653e-04	5.2591e-04
E_1(Dvt(k))	4.09612e-16	5.0685e-06	5.6535e-07	6.3079e-08	7.2094e-09	9.1851e-10	2.3570e-10
E_1(Opt(k))	4.09612e-16	2.3961e-17	1.6697e-17	3.0466e-17	1.1200e-16	5.0003e-16	2.0658e-15
E_2(Gen(k))	1.3636e-01	1.1870e-01	1.1399e-01	1.1280e-01	1.1250e-01	1.1242e-01	1.1240e-01
E_2(Hie(k))	1.3636e-01	5.9239e-02	3.6516e-02	3.0017e-02	2.7743e-02	2.6930e-02	2.6638e-02
E_2(Dvt(k))	1.3636e-01	5.9297e-02	4.1017e-02	3.8686e-02	3.8308e-02	3.8243e-02	3.8238e-02
E_2(Opt(k))	1.3636e-01	5.9294e-02	4.1014e-02	3.8685e-02	3.8308e-02	3.8243e-02	3.8238e-02

$$\mathrm{ME}_j(M) = \frac{1}{\sum_{p_i\in M} A_M(p_i)} \sum_{p_i\in M} A_M(p_i)E_j(M,p_i), \quad j=1,2.$$

The numerical results show that, though the maximal error bound E_2 is almost no-decreasing as k increase, for three types of meshes, ME_2 decreases to zero. This shows that at most of the points, the discrete LB operator converge. The errors E_2(Gen(5), p) and E_2(Hie(5), p) and E_2(Opt(5), p) are color-coded in Fig. 6.1 (the first row). The figure shows how the errors are distributed over the sphere. We further plot in Fig. 6.1 (the second row) the error function E_2 over the sphere. The exhibited triangular surfaces are defined by moving the vertex p_i of Gen(5), Hie(5) and Opt(5) in the outward normal direction at the distance E_2(Gen(5), p_i), E_2(Hie(5), p_i) and E_2(Opt(5), p_i), respectively. It is easy to observe that the error function E_2(Gen(k), p) is very bumpy, while E_2(Opt(k), p) is very smooth except for the region around the vertices with valence 5. In these regions, larger errors appear. E_2(Hie(5), p) is much more smooth than E_2(Gen(5), p), but less smooth than E_2(Opt(5), p).

Table 6.2: ME_1 and ME_2 for Gen(k), Hie(k), Dvt(k) and Opt(k)

k	1	2	3	4	5	6	7
ME_1(Gen(k))	1.1777e-16	1.3831e-02	6.4570e-03	2.1502e-03	6.2518e-04	1.6987e-04?	4.4493e-05
ME_1(Hie(k))	1.1777e-16	4.5070e-18	1.5972e-04	7.0219e-05	1.2019e-05	6.3252e-06	1.7660e-06
ME_1(Dvt(k))	1.1777e-16	1.9668e-06	1.5980e-07	1.1641e-08	7.8874e-10	5.2173e-11	4.5215e-12
ME_1(Opt(k))	1.1777e-16	1.4897e-18	2.3373e-18	8.1874e-18	3.1845e-17	1.2629e-16	5.0398e-16
ME_2(Gen(k))	8.6390e-02	4.6538e-02	1.8349e-02	6.2626e-03	1.9726e-03	5.9184e-04	1.7154e-04
ME_2(Hie(k))	8.6390e-02	2.8825e-02	9.0814e-03	2.7999e-03	1.0406e-03	4.7514e-04	2.4724e-04
ME_2(Dvt(k))	8.6390e-02	2.9376e-02	9.4659e-03	2.8831e-03	8.4399e-04	2.4014e-04	6.6938e-05
ME_2(Opt(k))	8.6390e-02	2.9375e-02	9.4656e-03	2.8831e-03	8.4399e-04	2.4014e-04	6.6938e-05

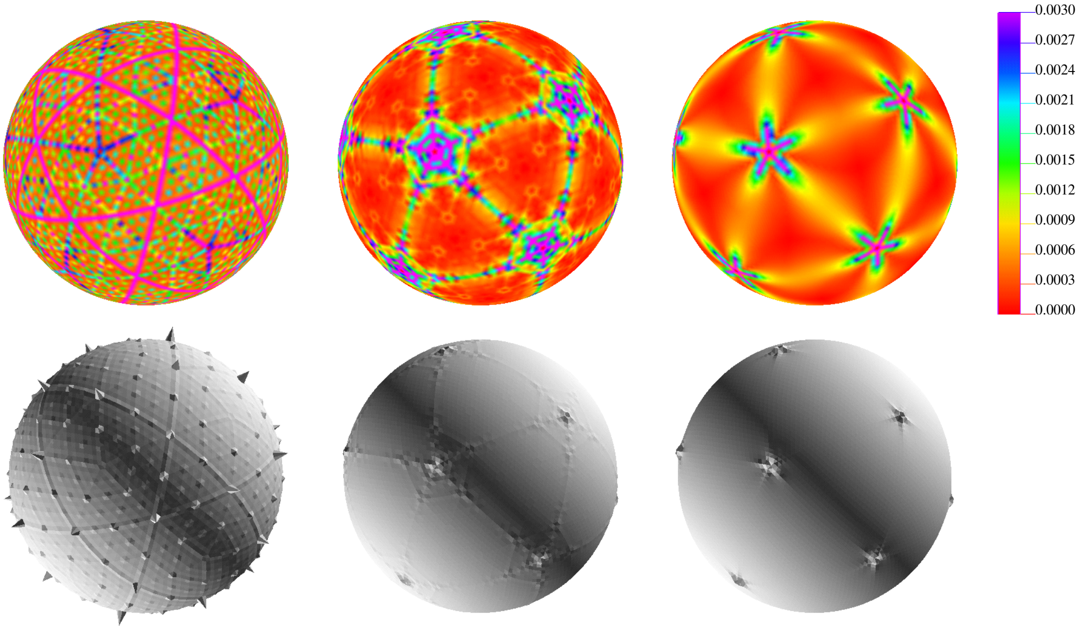

Fig 6.1: First row: Error E_2(Gen(5)) (the first), E_2(Hie(5)) (the second) and E_2(Opt(5)) (the third) are color-coded. The color map is shown on the right. Second row: Error functions E_2(Gen(5)), E_2(Hie(5)) and E_2(Opt(5)) are plotted as functions on spherical surface

7 Conclusion

We have shown that, for a sphere it is possible to construct triangulations such that the discrete mean curvature is exact. However, it may not possible to make discrete LB operator converge for any function over the sphere. Hence, we have sought such spherical triangulations that the truncations error is minimal. The constructed triangulations Opt(k) are optimal in the sense that they make the discrete mean curvature approximation exact and truncation error of the LB operator is minimal. Furthermore, the theoretical analysis has shown that the modification over the discrete LBO proposed by Meyer et al. is very significant, which makes the discrete mean curvature always converge.

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