数学与系统科学研究院

计算数学所学术报告

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报告题目：

Discontinuous Petrov Galerkin Method with Optimal Test Functions

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计算数学所报告厅

Abstract:

The concept of a variational formulation is usually attributed to Johann Bernoulli as it is directly linked to the classical Calculus of Variations started by Johann and Jacob Bernoulli and later developed by Euler and Lagrange. Indeed, mid-way between the minimization problem and the Euler-Lagrange equations, we arrive at an integral identity that has to be satisfied for all admissible variations, the Principle of Virtual Work. The essence of the principle is the fact that the solution is characterized through its action on test functions. In the mathematical language, we are dealing with an operator that takes values in the dual to the test space. Each of the three formulations: the minimization problem, the Euler-Lagrange equations, and the variational formulation may provide a starting point for a numerical approximation.

If the minimized functional is represented by a quadratic form, the corresponding variational problem is linear. If the quadratic form is positive definite (the functional is strictly convex), the minimization and variational problems are fully equivalent. This equivalence carries over to the discrete level and represents the essence of the Ritz method: solution of the discrete variational problem is equivalent to the minimization of discrete energy. This guarantees the stability of Finite
Element (FE) discretization regardless of a mesh being used. The Ritz method always delivers the best approximation in the sense of the energy norm.

If we focus on the equivalence of variational formulation and the Euler-Lagrange equations (based on integration by parts and Fourier's lemma), we realize that variational (weak) formulations can be developed for arbitrary problems described by Partial Differential Equations (PDEs). The essence of the Galerkin method is then to discretize the variational formulation rather than the PDEs. The critical question is whether the Galerkin method will converge beyond the ``safe scenarios'' provided by the positive definite self-adjoint operators (the Ritz setting).

A partial answer has been provided by Mikhlin's theory of asymptotic stability. If a positive definite self-adjoint operator is perturbed with a lower order term (compact operator), the Galerkin method is asymptotically stable and in fact optimal: for fine enough meshes, it will deliver again the best approximation error. To this class of problems belong for instance standard vibrations and wave propagation problems. The delicate issue is how to determine whether the mesh is fine enough to guarantee the stability...

A more fundamental idea was proposed by Petrov in 1959 who suggested using different test and trial spaces in the Galerkin formulation. If the trial space

should be used to guarantee the approximability of the solution, the main role of the test space is to provide stability. We arrive at the fundamental Babuska's Theorem (1971) and the concept of the inf-sup condition, rooted in Banach Closed Range Theorem: if the test space can be selected in such a way that the discrete inf-sup condition is satisfied, the method will be stable and converge. The famous phrase states: ``discrete stability and approximability imply convergence''.

The practical issue how to select the test space remains and, in essence, has been the main focus of all FE developments in the last four decades including mixed methods, stabilized methods, bubble methods, exact sequences, etc.

Jay Gopalakrishnan and I presented a new FE method that automatically guarantees discrete stability by means of a Petrov-Galerkin scheme with optimal test functions computed on a fly. The main idea is very simple: compute (approximately) and use test functions that realize the supremum in the inf-sup conditions - the best test functions you can have. Surprise or not, we arrive at a minimum residual method (generalized least squares) in which the approximate solution delivers again the best approximation error in a special ``energy'' (residual) norm. The circle has been closed - we are back to the Ritz setting but now for any class of linear problems.

Critical to the practicality of the method is the use of discontinuous test functions (``broken'' test spaces) and so-called ultra-weak variational formulation.

In collaboration with several colleagues, we managed to develop a general theory for linear Partial Differential Equations (PDEs) including singular perturbation problems. The methodology has been applied to a variety of usual model problems: Poisson equation, convection-diffusion, elasticity, wave propagation: acoustics, electromagnetics, elastodynamics, Stokes, beams and shells. It also has been formally extended to nonlinear problems and applied to both incompressible and compressible Navier-Stokes equations.

I will conclude my presentation by flashing a few representative numerical results.

欢迎大家参加！